# Refinable Vector Splines and Multi-wavelets with Shortest Matrix Filters

by

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# Abstract

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A widely used class of basis functions in signal analysis is obtained from the dilation and integer shifts of a given (compactly supported) wavelet $\psi : \mathbb{R} \to \mathbb{R}$, by means of which a (scalar) signal can be decomposed into its low frequency and high frequency components. Whereas initially much attention was devoted to orthogonal wavelet decomposition techniques (see for example [1] and [2]), the recent book [3] introduced a more general approach to wavelet construction in which orthogonality is not a requirement and which yielded significant advantages in some application areas.

An interesting extension is to consider instead, with the view to the decomposition of a vector-valued signal, as presented for the orthogonal case in, for example, [4], a multi-wavelet $\mathbf{\Psi} : \mathbb{R} \to \mathbb{R}^{\nu}$. The main focus of this study is to extend the methods in [3], in order to characterize, by means of matrix Laurent polynomial identity systems, a class of multi-wavelets based on general (not necessarily orthogonal) space decomposition.

As main building blocks are used refinable vector functions , together with their corresponding matrix refinement sequences. Three different classes of refinable vector splines are analysed, with particular focus also on their integer-shift linear independence and stability properties, before explicitly constructing their corresponding spline multi-wavelets. The low-pass and high-pass decomposition matrix filter sequences thus obtained are the shortest possible for the given refinable vector spline, and the spline multi-wavelet is of minimal support for these optimal matrix filters. Moreover, our approach yields explicit formulations for the refinable vector splines, as well as for their corresponding spline multi-wavelets and matrix filter sequences.

Computationally efficient algorithms are developed, and examples are calculated, with accompanying illustrating graphs.

# Uittreksel

## Verfynbare Vektorlatfunksies en Multi-golfies met Kortse Matriksfilters

*("Refinable Vector Splines and Multi-wavelets with Shortest Matrix Filters")*

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'n Wydgebruikte klas van basisfunksies in seinanalise word verkry uit die dilasie en heeltalskuiwe van 'n gegewe ( kompak-ondersteunde) golfie $\psi : \mathbb{R} \to \mathbb{R}$, deur middel waarvan 'n (skalaar-) sein in lae en hoë frekwensie komponente ontbind kan word. Waar daar aanvanklik baie aandag bestee is aan ortogonale golfie-ontbindingstegnieke ( sien byvoorbeeld [1] en [2]), het die onlangse boek [3] 'n meer algemene benadering bekendgestel waarin ortogonaliteit nie 'n vereiste is nie, en wat beduidende voordele in sommige toepassingsgebiede opgelewer het.

'n Interessante uitbreiding is om instede te beskou, met die oog op die ontbinding van 'n vektorsein, soos aangebied vir die ortogonale geval in, byvoorbeeld, [4], 'n multi-golfie $\boldsymbol{\Psi} : \mathbb{R} \to \mathbb{R}^\nu$. Die hooffokus van hierdie studie is om die metodes van [3] uit te brei, met die doel om, deur middel van matriks-Laurentpolinome, 'n klas multi-golfies gebaseer op algemene ( nie noodwendig ortogonale) ruimte-dekomposisie te karakteriseer.

As hoofboustene word gebruik verfynbare vektorfunksies, tesame met ooreenkomstige matriks-verfyningsrye. Drie verskillende klasse verfynbare vektorlatfunksies word ge-analiseer, met spesifieke aandag ook op hulle heeltalskuif lineêre onafhanklikheid- en stabiliteitseienskappe, voordat hulle ooreenkomstige latfunksie multi-golfies eksplisiet gekonstrueer word. Die lae-deurgang en hoëdeurgang ontbindings matriksfilterrye wat sodoende verkry word is die kortste moontlik vir die gegewe verfynbare vektorlatfunksie, en die latfunksie multi-golfie is van minimale steun vir hierdie optimale matriksfilters. Ons benadering lewer boonop eksplisiete formulerings vir die verfynbare vektorlatfunksies, asook vir hulle ooreenkomstige latfunksie multi-golfies en matriksfilterrye.

Berekeningsdoeltreffende algoritmes word ontwikkel, en voorbeelde word uitgewerk, met bygaande illustrerende grafieke.

# Acknowledgements

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# List of symbols

### Sets of numbers

| | |
|---|---|
| $\mathbb{Z}$ | Integers |
| $\mathbb{N}$ | Positive integers |
| $\mathbb{R}$ | Real numbers |
| $\mathbb{C}$ | Complex numbers |

### Function spaces

| | |
|---|---|
| $\pi_k$ | Polynomials of degree at most $k$ |
| $C(\mathbb{R})$ | Continuous (scalar) functions |
| $C_0(\mathbb{R})$ | Compactly supported (scalar) functions in $C(\mathbb{R})$ |
| $C^{-1}(\mathbb{R})$ | Piecewise continuous (scalar) functions |
| $C^l(\mathbb{R})$ | $l$-times continuously differentiable (scalar) functions |
| $C^\infty(\mathbb{R})$ | Infinitely differentiable (scalar) functions |
| $C_0^l(\mathbb{R})$ | $C_0(\mathbb{R}) \cap C^l(\mathbb{R})$ |
| $\mathbf{C}(\mathbb{R})$ | Vector functions with component functions in $C(\mathbb{R})$ |
| $\mathbf{C}_0(\mathbb{R})$ | Vector functions with component functions in $C_0(\mathbb{R})$ |
| $\mathbf{C}^{-1}(\mathbb{R})$ | Vector functions with component functions in $C^{-1}(\mathbb{R})$ |
| $\mathbf{C}^l(\mathbb{R})$ | Vector functions with component functions in $C^l(\mathbb{R})$ |
| $\mathbf{C}^\infty(\mathbb{R})$ | Vector functions with component functions in $C^\infty(\mathbb{R})$ |
| $\mathbf{C}_0^l(\mathbb{R})$ | $\mathbf{C}_0(\mathbb{R}) \cap \mathbf{C}^l(\mathbb{R})$ |
| $L^1(\mathbb{R})$ | Lebesgue integrable (scalar) functions on $\mathbb{R}$ |
| $L^2(\mathbb{R})$ | Lebesgue square-integrable (scalar) functions on $\mathbb{R}$ |

$S_{\nu,k}(\mathbb{Z})$ The cardinal spline space of degree $\nu$, and deficiency $k$ $\big($i.e. $S_{\nu,k}(\mathbb{Z}) \subset C^{\nu-1-k}(\mathbb{R})\big)$

**Matrix spaces**

$\mathcal{M}_\nu$ $\nu \times \nu$-matrices with real entries

$l^{\nu\times\nu}(\mathbb{Z})$ Bi-infinite sequences $\{M(k)\} = \{M(k) : k \in \mathbb{Z}\} \subset \mathcal{M}_\nu$

$l_0^{\nu\times\nu}(\mathbb{Z})$ Finitely supported sequences in $l^{\nu\times\nu}(\mathbb{Z})$

$l^2(\mathbb{Z})$ Square summable (scalar) sequences

$l_0^2(\mathbb{Z})$ Finitely supported (scalar) sequences in $l^2(\mathbb{Z})$

$\mathcal{S}^{\nu\times\nu}$ A subspace of $\mathcal{M}_\nu$, such that its elements have, except for in its first column, zero entries off its main diagonal

**Norms**

$\| f \|_{L^2(\mathbb{R})}$ $L^2$-norm

$\| \{c(k)\} \|_{l^2(\mathbb{R})}$ $l^2$-norm

**Operators**

$*$ The convolution operator for functions defined on a continuous domain

$\mathcal{F}f$ or $\widehat{f}$ Fourier transform of a Lebesgue integrable function $f$

$\mathcal{F}^{-1}f$ Inverse Fourier transform of $f$

$(\cdot)_+$ The truncated power function

**Notation**

$\operatorname{supp} f$ The support of the function $f$

$\{\delta(k)\}$ The Kronecker delta sequence in $l(\mathbb{Z})$

$\binom{m}{n}$ The binomial coefficient

$I$ The identity matrix in $\mathcal{M}_\nu$

$O$ The zero matrix in $\mathcal{M}_\nu$

$\mathbf{0}$ The zero vector in $\mathbb{R}^\nu$

$\det(M)$ The determinant of a matrix $M$

$M^{-1}$ The inverse of an invertible matrix $M$

# Chapter 1

# PRELIMINARIES

## 1.1 Introduction and overview

Wavelet decomposition is an important technique in signal analysis ([5], [6]), which was introduced to obtain a time-frequency localisation (see e.g. [2], [7]) in order to distinguish between the low frequency and the high frequency components of the signal, or to reconstruct the original signal if the previous information is given. Wavelet analysis can also be applied in other applications of data analysis such as imaging ([8], [9]) and data mining ([10]), as well as in other areas of mathematics such as computer graphics ([11], [12]), wavelet-based numerical analysis ([13], [14]), and more ([15], [16],[17]).

One method of constructing (scalar) wavelets in a continuous domain is by applying the so-called *multiresolution analysis* (see e.g. [5]), according to which one can consider a (scalar) function having the *d-refinability* property in order to generate a nested sequence of *refinement spaces* $\{S_r\}$, i.e. $S_r \subset S_{r+1}$, where each space $S_r$ corresponds to the resolution level $r \in \mathbb{Z}$ for the chosen integer *dilation* (or scale) $d \geq 2$. Moreover, (scalar) wavelets belong to the function space $W_0$ such that the space decomposition property

$$S_{r+1} = S_r \oplus W_r, \quad r \in \mathbb{Z}, \tag{1.1.1}$$

is satisfied. Initially, it was considered to be standard procedure (see e.g. [1], [5], [2], [18], [19]) to impose the orthogonal space decomposition condition ( i.e. $S_r \perp W_r$, $r \in \mathbb{Z}$), which then guarantees (1.1.1). In order to obtain wavelets with shorter supports, no orthogonality condition was imposed in [3] for $d = 2$, where only techniques in linear algebra and advanced calculus, and specifically no Fourier transforms, were applied to construct such minimally supported wavelets with corresponding shortest high-pass and low-pass filter sequences. In [20], the wavelet construction of [3] was extended, for the cardinal spline case, to any $d \geq 2$.

In the recent decades, the construction of multi-wavelets, for the decomposition of vector-valued data, has attracted the interest of many mathematicians, by virtue also of the fact that, unlike continuous scalar wavelets, multi-wavelets can possess both the properties of symmetry and orthogonality ([1], [21], [22]). Moreover, in some applications, the use of multi-wavelets can be more efficient than scalar wavelets ([23], [24], [25]). Given a *d-refinable* vector function of length $\nu \in \mathbb{N}$, the idea of multiresolution analysis for the multi-wavelet construction, which generates a corresponding vector function (multi-wavelet) of length $\nu$, is quite similar to the scalar wavelet construction method. Many such multi-wavelets have already been constructed by imposing the orthogonality (or bi-orthogonality) condition (see e.g. [26], [27], [4], [28], [29]).

In this thesis, a multi-wavelet construction for $d = 2$ will be presented as an extension of the (scalar) method introduced in [3]. In the rest of Chapter 1, after describing the notation, we define the notion of vector refinability (which, from now on, will always refer to 2-refinability), together with some desirable properties such as smoothness, integer-shift linear independence and $l^2$-stability on $\mathbb{R}$. A quest for a class of refinable vector functions having all these desirable properties, will comprise our work in Chapters 2 to 4.

First, in Chapter 2, an iterative convolution technique will be applied to generate a class of refinable vector splines with arbitrary smoothness, and where the support of these vector splines increases linearly with the order of smoothness. However, except for the non-continuous starting vector function, this class of refinable vector functions lacks both the properties of integer-shift linear independence, as well as $l^2$-stability, on $\mathbb{R}$, which compromises their usefulness in multi-wavelet construction, but leaves open their applicability in vector subdivision for curves, as has already been investigated for the case $\nu = 2$ in [30]. Regarding the starting refinable vector spline, this non-continuous vector function was previously considered in [31] and [30] for $\nu = 2$. Here, we introduce an extension of this starting vector function to general $\nu \in \mathbb{N}$ by means of a definition based on Bernstein polynomials, as well as an explicit formulation of its corresponding matrix refinement sequence.

Next, in Chapter 3, by extending the construction method introduced in [31], which yielded a class of refinable vector splines with one non-smooth component, we obtain a class of arbitrarily smooth refinable vector splines which do possess the desirable properties of integer-shift linear independence, as well as $l^2$-stability, on $\mathbb{R}$, and compute their corresponding matrix refinement sequences. In [31], the construction method used as starting point the Fourier transform formulations of the vector splines obtained there, with the consequence that the inverse Fourier transform was required to obtain explicit spline formulations, and was therefore not readily amenable towards obtaining general explicit formulations of these splines. Our approach here is to reverse this procedure by first deriving explicit formulations of our more general class of vector splines, and only then reverting to Fourier transforms to prove refinability, linear independence and stability, as well as to obtain the corresponding matrix refinement sequences.

In Chapter 4, as a third class of refinable vector splines, we consider the refinable Hermite vector splines with arbitrary length, on the interval $[-1, 1]$, which combines the properties of interpolation, symmetry and integer-shift linear independence, as well as $l^2$-stability, on $\mathbb{R}$ (see e.g. [32], [33], [34]). As our contribution to this topic, we derive here explicit and recursive formulations of these vector splines, and we develop an algorithm for the computation of the corresponding matrix refinement sequences.

Finally, in Chapter 5, we focus on the main interest of this thesis, which is the construction of multi-wavelets as an extension of the (scalar) method in [3], where no orthogonality condition was imposed, and we only use methods of algebra and advanced calculus, and in particular no Fourier transforms. As we will see, our general multi-wavelet construction method introduced in Section 5.1 depends on solving a system of matrix Laurent polynomial identities. An immediate complicating factor which arises, is that these matrix Laurent polynomials have matrix coefficients, in which context it should always be kept in mind that matrix multiplication is non-commutative. In the subsequent Sections 5.2 and 5.3, we then proceed to solve the matrix Laurent polynomial identities of Section 5.1 for, respectively, the refinable vector splines of Chapters 4 and 3. In both Sections 5.2 and 5.3, our method yields shortest possible matrix decomposition filter sequences for the given refinable vector spline, and our multi-wavelet is minimally supported with respect to these optimally short matrix decomposition filters. In particular, our Hermite spline multi-wavelets of Section 5.2 has a smaller support than the analogous spline multi-wavelets constructed in [35], which is based on level-dependent (nonstationary) multiresolution analyses of $L^2(\mathbb{R})$, and also compared to the ones introduced in [33], where the so-called CBC method was applied.

In the final Section 5.3, we furthermore develop results in polynomial algebra to explicitly obtain optimally local multi-wavelet decomposition results. In fact, the refinable vector splines of Chapter 3 seem to be a natural extension to the vector setting of (scalar) cardinal splines, and we believe that our work in Section 5.3 contributes to the advancement of the general theory thereof, particularly in view of our explicit formulations of the refinable vector spline itself, as well as of its corresponding minimally supported multi-wavelets and shortest decomposition matrix filters.

## 1.2 Notation

We shall denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of real numbers by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$.

Let $\nu \in \mathbb{N}$. We write $\mathcal{M}_\nu$ for the space of all $\nu \times \nu$ matrices with real entries. Observe that we may write $\mathbb{R}$ for $\mathcal{M}_1$. The matrices $I, O \in \mathcal{M}_\nu$ are, respectively, the identity matrix and the zero matrix.

We use the symbol $l^{\nu\times\nu}(\mathbb{Z})$ to denote the space of all bi-infinite sequences $\{M(k)\} = \{M(k) : k \in \mathbb{Z}\} \subset \mathcal{M}_\nu$, whereas $l_0^{\nu\times\nu}(\mathbb{Z})$ denotes the subspace of $l^{\nu\times\nu}(\mathbb{Z})$ consisting of finitely supported sequences, that is, $\{M(k)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$ if and only if $M(k)$ is not the zero matrix in $\mathcal{M}_\nu$ for only a finite number of indices $k$. In the scalar case $\nu = 1$, we shall write $l(\mathbb{Z})$ for $l^{1\times1}(\mathbb{Z})$ and $l_0(\mathbb{Z})$ for $l_0^{1\times1}(\mathbb{Z})$.

Moreover, for any non-negative integer $k$, we shall write $\pi_k$ for the space of polynomials of degree at most $k$. Also, for any non-negative integer $j$, we shall adopt the binomial coefficient notation

$$\binom{j}{i} := \begin{cases} \dfrac{j!}{i!(j-i)!}, & i = 0, \cdots, j; \\ 0, & i \in \mathbb{Z} \setminus \{0, \cdots, j\}, \end{cases} \tag{1.2.1}$$

with the convention $0! := 1$.

We shall say that a vector function $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T : \mathbb{R} \to \mathbb{R}^\nu$, with length $\nu$, is compactly supported if all the functions in the set $\{\phi_k : k = 1, \cdots, \nu\}$ vanish identically outside some closed and bounded interval on $\mathbb{R}$, that is, the support interval supp $\phi_k$ is a bounded interval in $\mathbb{R}$ for each $k \in \{1, \cdots, \nu\}$. In the scalar case $\nu = 1$, we shall write $\phi$ for $\mathbf{\Phi} = (\phi_1)$.

The symbol $\mathbf{C}(\mathbb{R})$ will denote the space of vector functions $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T$ such that all the component functions in the set $\{\phi_k : \ k = 1, \cdots, \nu\}$ belong to the space $C(\mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Also, we use the symbol $\mathbf{C}_0(\mathbb{R})$ for the subspace of $\mathbf{C}(\mathbb{R})$ consisting of compactly supported vector functions. Furthermore, for any given integer $k \in \mathbb{N}$, we define $\mathbf{C}^k(\mathbb{R})$ as the subspace of $\mathbf{C}(\mathbb{R})$ consisting of vector functions $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T$ such that all the functions in the set $\{\phi_k : \ k = 1, \cdots, \nu\}$ belong to the space $C^k(\mathbb{R})$, the space of functions with $k$ continuous derivatives on $\mathbb{R}$. Moreover, we use the symbol $\mathbf{C}_0^k(\mathbb{R})$ for the subspace of $\mathbf{C}^k(\mathbb{R})$ consisting of compactly supported vector functions; and we also define $\mathbf{C}^0(\mathbb{R}) := \mathbf{C}(\mathbb{R})$ and $\mathbf{C}_0^0(\mathbb{R}) := \mathbf{C}_0(\mathbb{R})$. A function $\mathcal{F} : \mathbb{C} \setminus \{0\} \to \mathcal{M}_\nu$ defined by

$$\mathcal{F}(z) := \sum_{k=\alpha}^{\beta} M(k) z^k, \tag{1.2.2}$$

with matrix coefficients $\{M(k) : \ k = \alpha, \cdots, \beta\} \subset \mathcal{M}_\nu$, where $\alpha, \beta \in \mathbb{Z}$, with $\alpha \leq \beta$, will be called a matrix Laurent polynomial. If, moreover, $\alpha \geq 0$ in (1.2.2), we shall call $\mathcal{F}$ a matrix polynomial.

We shall denote by $\{\delta(k)\}$ the Kronecker delta sequence in $l(\mathbb{Z})$, as defined by

$$\delta(k) := \begin{cases} 1, & k = 0; \\ 0, & k \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{1.2.3}$$

We write $\sum\limits_k$ for $\sum\limits_{k \in \mathbb{Z}}$.

## 1.3 Refinability of vector functions

Let $\mathbf{\Phi}$ be a compactly supported vector function of length $\nu \in \mathbb{N}$. We shall say that $\mathbf{\Phi}$ is *refinable* if there exists a matrix sequence $\{P(k)\} \in l_0^{\nu \times \nu}(\mathbb{Z})$ such that

$$\mathbf{\Phi}(x) = \sum_k P(k) \mathbf{\Phi}(2x - k), \quad x \in \mathbb{R}, \tag{1.3.1}$$

in which case the identity (1.3.1) is called the *vector refinement equation*, whereas the sequence $\{P(k)\}$ is called the *matrix refinement sequence*, or *matrix mask*, that corresponds to the refinable vector function $\mathbf{\Phi}$. Also, the matrix Laurent polynomial $\mathcal{P}$ defined by

$$\mathcal{P}(z) := \frac{1}{2}\sum_{k} P(k)\; z^k, \quad z \in \mathbb{C} \setminus \{0\}, \tag{1.3.2}$$

will be called *matrix refinement symbol* of $\mathbf{\Phi}$.

For the scalar case $\nu = 1$, well-known examples of scalar refinable functions $\phi$ are provided by the cardinal B-splines $N_n$ of degree $n \in \mathbb{N} \cup \{0\}$, as defined recursively by means of convolution by

$$N_0 := \chi_{[0,1)}; \tag{1.3.3}$$

$$N_{n+1}(x) := \big(N_n * \chi_{[0,1)}\big)(x) = \int_0^1 N_n(x-t)\; dt = \int_{x-1}^{x} N_n(t)\; dt, \quad x \in \mathbb{R},\; n = 0, 1, \cdots, \tag{1.3.4}$$

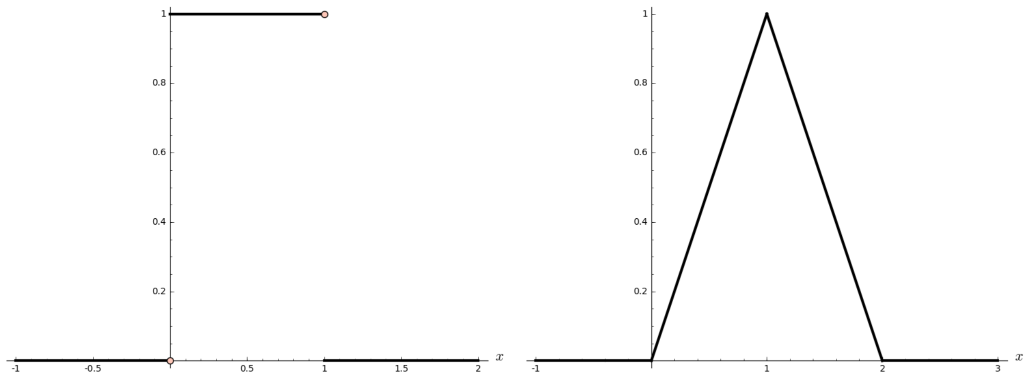
with, for any $A \subset \mathbb{R}$, the characteristic function of $A$ defined by

$$\chi_A(x) := \begin{cases} 1, & x \in A; \\ 0, & x \in \mathbb{R} \setminus A. \end{cases} \tag{1.3.5}$$

It follows from (1.3.4) that $N_1$ is the shifted hat function

$$N_1(x) := \begin{cases} x, & x \in [0,1); \\ 2-x, & x \in [1,2); \\ 0, & x \in \mathbb{R} \setminus [0,2). \end{cases} \tag{1.3.6}$$

The graphs of the cardinal B-splines $N_0$ and $N_1$ are given by Fig. 1.1.




**(a)** $N_0$ **(b)** $N_1$

**Figure 1.1:** *The cardinal B-splines* $N_0(=\chi_{[0,1)})$ *and* $N_1$

For any degree $n \in \mathbb{N} \cup \{0\}$, it holds (see e.g. [3]) that

$$N_n(x) = \frac{1}{2^n}\sum_{k} \binom{n+1}{k} N_n(2x-k), \quad x \in \mathbb{R}, \tag{1.3.7}$$

that is, $N_n$ is a refinable (scalar) function with (scalar) refinement sequence $\{p(k)\} = \{p_n(k)\}$ given by

$$p_n(k) := \frac{1}{2^n}\binom{n+1}{k}, \quad k \in \mathbb{Z}. \tag{1.3.8}$$

Observe from (1.3.8) that the corresponding (scalar) refinement sequences of the cardinal B-splines $N_0$ and $N_1$ are given by, respectively, $\{p_0(k)\}$ and $\{p_1(k)\}$, with

$$p_0(k) = \begin{cases} 1, & k = 0; \\ 1, & k = 1; \\ 0, & k \in \mathbb{Z} \setminus \{0, 1\}; \end{cases} \quad ; \quad p_1(k) = \begin{cases} \frac{1}{2}, & k = 0; \\ 1, & k = 1; \\ \frac{1}{2}, & k = 2; \\ 0, & k \in \mathbb{Z} \setminus \{0, 1, 2\}, \end{cases} \tag{1.3.9}$$

according to which the corresponding (scalar) refinement symbols are given by, respectively, $\mathcal{P}_0$ and $\mathcal{P}_1$, where

$$\mathcal{P}_0(z) = \frac{1+z}{2}, \qquad \mathcal{P}_1(z) = \left(\frac{1+z}{2}\right)^2. \tag{1.3.10}$$

In general, from (1.3.2) and (1.3.8), and for any $n \in \mathbb{N} \cup \{0\}$, the refinement symbol $\mathcal{P}_n$ corresponding to the cardinal B-spline $N_n$, is given by

$$\mathcal{P}_n(z) = \left(\frac{1+z}{2}\right)^{n+1}. \tag{1.3.11}$$

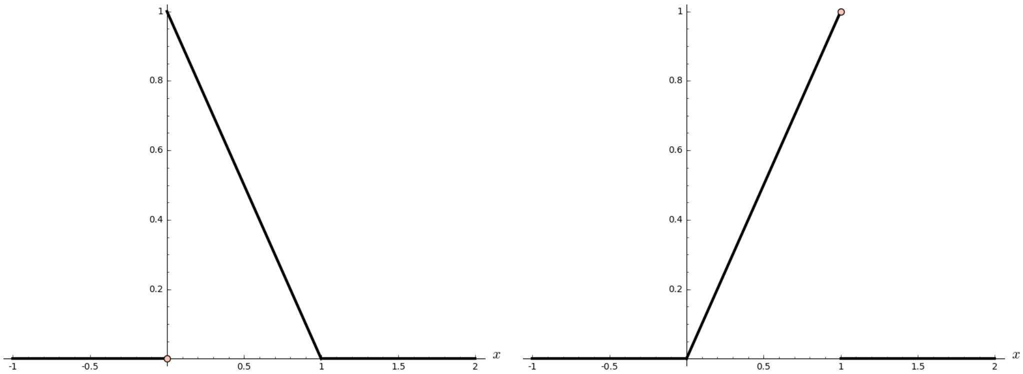
Note from (1.3.3) and (1.3.4) that, for $n \in \mathbb{N}$, the cardinal B-spline $N_n$ is a compactly supported piecewise polynomial of degree $n$, with respect to the integer partition $\mathbb{Z}$ of $\mathbb{R}$, with supp $N_n = [0, n+1]$, and where also, from the Fundamental Theorem of Calculus, $N_n \in C^{n-1}(\mathbb{R})$.

Next, for the vector case $\nu = 2$, we proceed to give two examples of refinable vector functions.

**Example 1.3.1** Let the vector function $\mathbf{\Phi} = (\phi_1, \phi_2)^T : \mathbb{R} \to \mathbb{R}^2$ be defined by (see e.g. [31], [30])

$$\phi_1(x) := \begin{cases} 1 - x, & x \in [0, 1); \\ 0, & x \in \mathbb{R} \setminus [0, 1); \end{cases} \qquad \phi_2(x) := \begin{cases} x, & x \in [0, 1); \\ 0, & x \in \mathbb{R} \setminus [0, 1), \end{cases} \tag{1.3.12}$$

for which the graphs are given in Fig. 1.2.




**(a)** $\phi_1$ **(b)** $\phi_2$

**Figure 1.2:** *The vector function* $\mathbf{\Phi} = (\phi_1, \phi_2)^T$, *as in* (1.3.12)

We proceed to show that $\mathbf{\Phi}$ is a refinable vector function with corresponding matrix refinement sequence $\{P(k)\} \in l_0^{2\times 2}(\mathbb{Z})$ given by

$$P(0) := \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}; \quad P(1) := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}; \quad P(k) := O, \quad k \in \mathbb{Z} \setminus \{0,1\}. \tag{1.3.13}$$

Let $x \in [0, \frac{1}{2})$, for which (1.3.12) gives

$$\left.\begin{aligned} &\phi_1(2x) = 1 - 2x; \quad \phi_1(2x-1) = 0; \\ &\phi_2(2x) = 2x; \qquad\;\; \phi_2(2x-1) = 0. \end{aligned}\right\} \tag{1.3.14}$$

It follows from (1.3.13) and (1.3.14) that

$$\sum_k P(k)\mathbf{\Phi}(2x-k) = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-2x \\ 2x \end{bmatrix} = \begin{bmatrix} 1-x \\ x \end{bmatrix} = \mathbf{\Phi}(x),$$

according to which the vector refinement equation (1.3.1) is satisfied for $x \in [0, \frac{1}{2})$. Next, for $x \in [\frac{1}{2}, 1)$, we see from (1.3.12) that

$$\left.\begin{aligned} &\phi_1(2x) = 0; \quad \phi_1(2x-1) = 2 - 2x, \\ &\phi_2(2x) = 0; \quad \phi_2(2x-1) = 2x - 1. \end{aligned}\right\} \tag{1.3.15}$$

It follows from (1.3.13) and (1.3.15) that

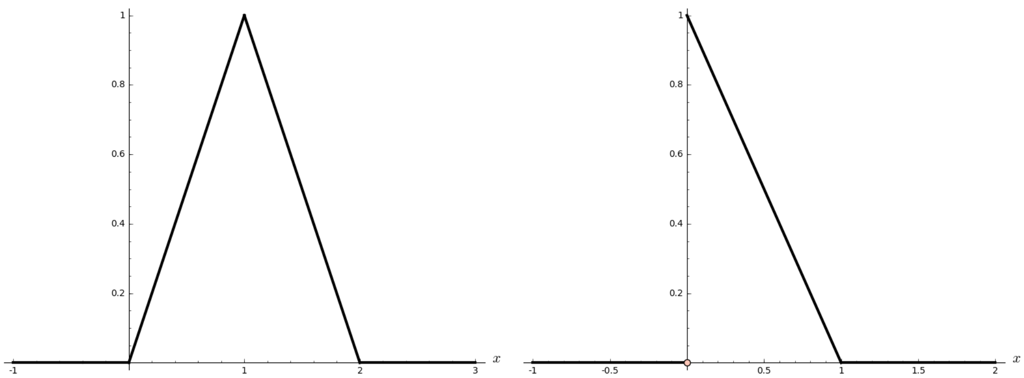
$$\sum_k P(k)\mathbf{\Phi}(2x-k) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2-2x \\ 2x-1 \end{bmatrix} = \begin{bmatrix} 1-x \\ x \end{bmatrix} = \mathbf{\Phi}(x),$$

according to which the vector refinement equation (1.3.1) is also satisfied for $x \in [\frac{1}{2}, 1)$. For $x \in \mathbb{R} \setminus [0,1)$, the vector refinement equation (1.3.1) is trivially satisfied with both sides equal to $\mathbf{0}$. Hence the vector refinement equation (1.3.1) is satisfied for all $x \in \mathbb{R}$, that is, $\mathbf{\Phi}$ is a refinable vector function with matrix refinement sequence $\{P(k)\}$ given as in (1.3.13). ■

**Example 1.3.2** Next, let the vector function $\mathbf{\Phi} = (\phi_1, \phi_2)^T$ be given by (see e.g. [31])

$$\phi_1(x) := N_1(x), \quad \phi_2(x) := \begin{cases} 1 - x, & x \in [0,1); \\ 0, & x \in \mathbb{R} \setminus [0,1), \end{cases} \tag{1.3.16}$$

where $N_1$ is the shifted hat function, as defined in (1.3.6), and with corresponding graphs given in Fig. 1.3.




**(a)** $\phi_1$ **(b)** $\phi_2$

**Figure 1.3:** *The vector function* $\mathbf{\Phi} = (\phi_1, \phi_2)^T$*, as in* (1.3.16)

We proceed to show that $\mathbf{\Phi}$ is a refinable vector function with corresponding matrix refinement sequence $\{P(k)\} \in l_0^{2\times 2}(\mathbb{Z})$ given by

$$P(0) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}; \quad P(1) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}; \quad P(2) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \tag{1.3.17}$$
$$P(k) = O, \quad k \in \mathbb{Z} \setminus \{0,1,2\}.$$

Let $x \in [0, \frac{1}{2})$, for which (1.3.6) and (1.3.16) give

$$\left.\begin{aligned} \phi_1(2x) &= 2x; & \phi_1(2x-1) &= 0; & \phi_1(2x-2) &= 0; \\ \phi_2(2x) &= 1 - 2x; & \phi_2(2x-1) &= 0; & \phi_2(2x-2) &= 0. \end{aligned}\right\} \tag{1.3.18}$$

It follows from (1.3.17), (1.3.18) and (1.3.16) that

$$\begin{aligned} \sum_k P(k)\mathbf{\Phi}(2x-k) &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2x \\ 1-2x \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x \\ 1-x \end{bmatrix} = \mathbf{\Phi}(x), \end{aligned}$$

according to which the vector refinement equation (1.3.1) is satisfied for $x \in [0, 1/2)$.

For $x \in [\frac{1}{2}, 1)$, it follows from (1.3.6) and (1.3.16) that

$$\left.\begin{array}{llll} \phi_1(2x) & = 2-2x; & \phi_1(2x-1) = 2x-1; & \phi_1(2x-2) = 0; \\ \phi_2(2x) & = 0; & \phi_2(2x-1) = 2-2x; & \phi_2(2x-2) = 0. \end{array}\right\} \tag{1.3.19}$$

It follows from (1.3.17), (1.3.19) and (1.3.16) that

$$\begin{aligned} \sum_k P(k)\mathbf{\Phi}(2x-k) &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2-2x \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2x-1 \\ 2-2x \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1-x \\ 1-x \end{bmatrix} + \begin{bmatrix} 2x-1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix} = \mathbf{\Phi}(x), \end{aligned}$$

according to which the vector refinement equation (1.3.1) is satisfied for $x \in [\frac{1}{2}, 1)$.

For $x \in [1, \frac{3}{2})$, it follows from (1.3.6) and (1.3.16) that

$$\left.\begin{array}{llll} \phi_1(2x) & = 0; & \phi_1(2x-1) = 3-2x; & \phi_1(2x-2) = 2x-2; \\ \phi_2(2x) & = 0; & \phi_2(2x-1) = 0; & \phi_2(2x-2) = 3-2x. \end{array}\right\} \tag{1.3.20}$$

It follows from (1.3.17), (1.3.20) and (1.3.16) that

$$\begin{aligned} \sum_k P(k)\mathbf{\Phi}(2x-k) &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3-2x \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2x-2 \\ 3-2x \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3-2x \\ 0 \end{bmatrix} + \begin{bmatrix} x-1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-x \\ 0 \end{bmatrix} = \mathbf{\Phi}(x), \end{aligned}$$

according to which the vector refinement equation (1.3.1) is satisfied for $x \in [1, \frac{3}{2})$.

Let $x \in [\frac{3}{2}, 2)$, for which (1.3.6) and (1.3.16) give

$$\left.\begin{array}{llll} \phi_1(2x) & = 0; & \phi_1(2x-1) = 0; & \phi_1(2x-2) = 4-2x; \\ \phi_2(2x) & = 0; & \phi_2(2x-1) = 0; & \phi_2(2x-2) = 0. \end{array}\right\} \tag{1.3.21}$$

It follows from (1.3.17), (1.3.21) and (1.3.16) that

$$\begin{aligned} \sum_k P(k)\mathbf{\Phi}(2x-k) &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4-2x \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2-x \\ 0 \end{bmatrix} = \mathbf{\Phi}(x), \end{aligned}$$

according to which the vector refinement equation (1.3.1) is satisfied for $x \in [\frac{3}{2}, 2)$.

For $x \in \mathbb{R} \setminus [0, 2)$, the vector refinement equation (1.3.1) is trivially satisfied with both sides equal to $\mathbf{0}$. We have therefore now shown that the vector refinement equation (1.3.1) is satisfied for all $x \in \mathbb{R}$, and thus $\mathbf{\Phi}$ is indeed a vector refinable function with matrix refinement sequence $\{P(k)\}$ given by (1.3.17). ■

## 1.4 Integer-shift linear independence

For any integer $\nu \in \mathbb{N}$, if a compactly supported vector function $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T$ : $\mathbb{R} \to \mathbb{R}^\nu$ has the property that the only matrix sequence $\{M(k)\} \in l^{\nu \times \nu}(\mathbb{Z})$ satisfying the identity

$$\sum_k M(k)\mathbf{\Phi}(x-k) = \mathbf{0}, \quad x \in \mathbb{R}, \tag{1.4.1}$$

is the zero matrix sequence, that is,

$$M(k) = O, \quad k \in \mathbb{Z}, \tag{1.4.2}$$

we say that $\mathbf{\Phi}$ possesses *matrix linearly independent integer shifts on* $\mathbb{R}$. Observe that (1.4.1) implies (1.4.2) if and only if the only sequences $\{a_1(k)\}, \cdots, \{a_\nu(k)\} \in l(\mathbb{Z})$ satisfying the identity

$$\sum_{j=1}^{\nu} \sum_k a_j(k)\phi_j(x-k) = 0, \quad x \in \mathbb{R}, \tag{1.4.3}$$

are the zero sequences, that is,

$$a_j(k) = 0, \quad k \in \mathbb{Z}; \quad j = 1, \cdots, \nu. \tag{1.4.4}$$

For the scalar case $\nu = 1$, it is known ([3]) that, for any non-negative integer $n$, the cardinal B-spline $N_n$ possesses linearly independent integer shifts on $\mathbb{R}$.

For $\nu = 2$, we proceed to investigate the integer-shift linear independence of the vector functions $\mathbf{\Phi}$ in, respectively, Examples 1.3.1 and 1.3.2.

**Example 1.4.1** Let $\mathbf{\Phi} = (\phi_1, \phi_2)^T$ denote the vector function defined in (1.3.12) of Example 1.3.1, and suppose $\{a_1(k)\}$ and $\{a_2(k)\}$ are two sequences in $l(\mathbb{Z})$ such that

$$\sum_k a_1(k)\phi_1(x-k) + \sum_k a_2(k)\phi_2(x-k) = 0, \quad x \in \mathbb{R}. \tag{1.4.5}$$

Observe from (1.4.5) that

$$\sum_k a_1(k)\phi_1(j-k) + \sum_k a_2(k)\phi_2(j-k) = 0, \quad j \in \mathbb{Z}. \tag{1.4.6}$$

Since the definition (1.3.12) yields

$$\left.\begin{aligned} \phi_1(j) &= \delta(j); \\ \phi_2(j) &= 0, \end{aligned}\right\} \quad j \in \mathbb{Z}, \tag{1.4.7}$$

it follows from (1.4.6) that

$$a_1(j) = 0, \quad j \in \mathbb{Z}, \tag{1.4.8}$$

which may now be substituted into (1.4.5) to obtain

$$\sum_k a_2(k)\phi_2(x-k) = 0, \quad x \in \mathbb{R}. \tag{1.4.9}$$

For any $j \in \mathbb{Z}$, let $x \in [j, j+1)$. It then follows from (1.3.12) that

$$\sum_k a_2(k)\phi_2(x-k) = a_2(j)(x-j). \tag{1.4.10}$$

Hence, from (1.4.9) and (1.4.10),

$$a_2(j)(x-j) = 0, \quad x \in [j, j+1),$$

which implies $a_2(j) = 0$, and thus, since $j \in \mathbb{Z}$ was arbitrarily chosen,

$$a_2(j) = 0, \quad j \in \mathbb{Z}. \tag{1.4.11}$$

Hence we have shown that the only sequences $\{a_1(k)\}$ and $\{a_2(k)\}$ in $l(\mathbb{Z})$ satisfying the identity (1.4.5) are the zero sequences, that is,

$$a_j(k) = 0, \quad k \in \mathbb{Z}; \quad j = 1, 2, \tag{1.4.12}$$

which shows that the vector function $\mathbf{\Phi}$ possesses matrix linearly independent integer shifts on $\mathbb{R}$. ■

**Example 1.4.2** Let $\mathbf{\Phi} = (\phi_1, \phi_2)^T$ denote the vector function defined in (1.3.16) of Example 1.3.2, and suppose $\{a_1(k)\}$ and $\{a_2(k)\}$ are two sequences in $l(\mathbb{Z})$ such that

$$\sum_k a_1(k)\phi_1(x-k) + \sum_k a_2(k)\phi_2(x-k) = 0, \quad x \in \mathbb{R}, \tag{1.4.13}$$

and thus

$$\sum_k a_1(k)\phi_1(j-k) + \sum_k a_2(k)\phi_2(j-k) = 0, \quad j \in \mathbb{Z}. \tag{1.4.14}$$

Since the definition (1.3.16), with $N_1$ denoting the shifted hat function as in (1.3.6), yields

$$\left.\begin{aligned} &\phi_1(j-1) = \delta(j); \\ &\phi_2(j) = \delta(j), \end{aligned}\right\} \quad j \in \mathbb{Z}, \tag{1.4.15}$$

we deduce from (1.4.14) that

$$a_1(j-1) + a_2(j) = 0, \quad j \in \mathbb{Z},$$

that is,

$$a_2(j) = -a_1(j-1), \quad j \in \mathbb{Z}, \tag{1.4.16}$$

which we may now insert into (1.4.13) to obtain, for any $x \in \mathbb{R}$,

$$\begin{aligned} 0 &= \sum_k a_1(k)\phi_1(x-k) - \sum_k a_1(k-1)\phi_2(x-k) \\ &= \sum_k a_1(k)\phi_1(x-k) - \sum_k a_1(k)\phi_2(x-k-1) \\ &= \sum_k a_1(k)[\phi_1(x-k) - \phi_2(x-k-1)], \end{aligned}$$

and thus

$$\sum_k a_1(k)\widetilde{\phi}(x-k) = 0, \quad x \in \mathbb{R}, \tag{1.4.17}$$

where

$$\widetilde{\phi}(x) := \phi_1(x) - \phi_2(x-1). \tag{1.4.18}$$

It then follows from (1.4.18) and (1.3.16), (1.3.6) that

$$\widetilde{\phi}(x) = \begin{cases} x, & x \in [0,1); \\ 0, & x \in \mathbb{R} \setminus [0,1), \end{cases} \tag{1.4.19}$$

according to which

$$\text{supp } \tilde{\phi} = [0,1]. \tag{1.4.20}$$

For any $j \in \mathbb{Z}$, let $x \in [j, j+1)$. It then follows from (1.4.19) that

$$\sum_k a_1(k)\tilde{\phi}(x-k) = a_1(j)\ (x-j). \tag{1.4.21}$$

Hence, from (1.4.17) and (1.4.21),

$$a_1(j)\ (x-j) = 0, \quad x \in [j, j+1),$$

which implies $a_1(j) = 0$, and thus, since $j \in \mathbb{Z}$ was arbitrarily chosen,

$$a_1(j) = 0, \quad j \in \mathbb{Z}. \tag{1.4.22}$$

It then follows from (1.4.16) and (1.4.22) that also

$$a_2(j) = 0, \quad j \in \mathbb{Z}. \tag{1.4.23}$$

Hence we have shown that the only sequences $\{a_1(k)\}$ and $\{a_2(k)\}$ in $l(\mathbb{Z})$ satisfying the identity (1.4.13) are the zero sequences, that is,

$$a_j(k) = 0, \quad k \in \mathbb{Z}; \quad j = 1, 2, \tag{1.4.24}$$

which shows that the vector function $\boldsymbol{\Phi}$ possesses matrix linearly independent integer shifts on $\mathbb{R}$. ■

## 1.5 Integer-shift $l^2$-stability

In this section, we introduce the concept of $l^2$-stability for the integer shifts of a compactly supported vector function, as is often of essential importance in numerical applications.

We write $l^2(\mathbb{Z})$ for the subspace of $l(\mathbb{Z})$ consisting of square summable sequences, that is,

$$l^2(\mathbb{Z}) := \Big\{\{c(k)\} \in l(\mathbb{Z}); \quad \sum_k \big[c(k)\big]^2 < \infty\Big\}, \tag{1.5.1}$$

for which we define the norm

$$\|\ \{c(k)\}\ \|_{l^2(\mathbb{Z})} := \sqrt{\sum_k [c(k)]^2}, \qquad \{c(k)\} \in l^2(\mathbb{Z}). \tag{1.5.2}$$

Also, we denote by $L^2(\mathbb{R})$ the space of Lebesgue square-integrable real-valued functions on $\mathbb{R}$, that is,

$$L^2(\mathbb{R}) := \Big\{f : \mathbb{R} \longrightarrow \mathbb{R}; \quad \int_{-\infty}^{\infty} [f(x)]^2\ dx < \infty\}, \tag{1.5.3}$$

for which we define the norm

$$\| f \|_{L^2(\mathbb{R})} := \sqrt{\int_{-\infty}^{\infty} [f(x)]^2 \, dx}, \quad f \in L^2(\mathbb{R}). \tag{1.5.4}$$

We shall rely on the following standard result (see e.g. [36, Theorem 2.1]).

**Theorem 1.5.1** *For any function $f \in L^2(\mathbb{R})$ and sequence $\{a(k)\} \in l^2(\mathbb{Z})$, the definition*

$$g(x) := \sum_k a(k) f(x-k), \quad x \in \mathbb{R}, \tag{1.5.5}$$

*yields a function $g \in L^2(\mathbb{R})$, with*

$$\| g \|_{L^2(\mathbb{R})} \leq \| \{a(k)\} \|_{l^2(\mathbb{Z})} \| f \|_{L^2(\mathbb{R})} . \tag{1.5.6}$$

For any $\nu \in \mathbb{N}$, let $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T : \mathbb{R} \to \mathbb{R}^\nu$ denote a vector function satisfying the condition

$$\phi_k \in L^2(\mathbb{R}), \quad k = 1, \cdots, \nu. \tag{1.5.7}$$

If, moreover, there exist positive constants $A$ and $B$ such that

$$A \sum_{j=1}^{\nu} \| \{a_j(k)\} \|_{l^2(\mathbb{Z})} \leq \| \sum_{j=1}^{\nu} \sum_k a_j(k) \phi_j(\cdot - k) \|_{L^2(\mathbb{R})} \leq B \sum_{j=1}^{\nu} \| \{a_j(k)\} \|_{l^2(\mathbb{Z})},$$
$$\{a_j(k)\} \in l^2(\mathbb{Z}), \quad j = 1, \cdots, \nu, \tag{1.5.8}$$

we say that $\mathbf{\Phi}$ possesses $l^2$-stable integer shifts on $\mathbb{R}$. Observe that the second inequality in (1.5.8) is automatically satisfied as an immediate consequence of Theorem 1.5.1, according to which we may choose

$$B = \max\{ \| \phi_1 \|_{L^2(\mathbb{R})}, \cdots, \| \phi_\nu \|_{L^2(\mathbb{R})} \}. \tag{1.5.9}$$

As proved in [36, Sections 4 and 5] (see also [37]), linear independence implies $l^2$-stability, as follows.

**Theorem 1.5.2** *For any integer $\nu \in \mathbb{N}$, let $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T$ denote a vector function with matrix linearly independent integer shifts on $\mathbb{R}$, and such that the condition* (1.5.7) *is satisfied. Then $\mathbf{\Phi}$ possesses $l^2$-stable integer shifts on $\mathbb{R}$.*

We may now apply Theorem 1.5.2 to deduce that each of the two refinable vector splines $\mathbf{\Phi}$ in, respectively, Examples 1.4.1 and 1.4.2 possesses $l^2$-stable integer shifts on $\mathbb{R}$.

In this chapter, we introduced the notion of vector refinability in Section 1.3, which is also called *self-similarity*. From this definition, a refinable vector function is then characterized by its corresponding matrix refinement sequence to satisfy the vector refinement equation (1.3.1). Furthermore, the properties of integer-shift linear independence, as well as the integer-shift $l^2$-stability, on $\mathbb{R}$ have been highlighted in Section 1.4, and in which we pointed out in Theorem 1.5.2 that linear independence implies $l^2$-stability. The two refinable vector functions of length 2 in, respectively, Examples 1.3.1 and 1.3.2, although possessing these desirable properties, are not continuous. In the next two chapters, we will establish arbitrarily smooth extensions of, respectively, the refinable vector functions (1.3.12) and (1.3.16), and for any length $\nu \in \mathbb{N}$.

# Chapter 2

# SMOOTH REFINABILITY FROM CONVOLUTION

For both the refinable vector functions $\mathbf{\Phi}$ in (1.3.12) and (1.3.16), we have $\mathbf{\Phi} \notin \mathbf{C}(\mathbb{R})$. In this chapter, we introduce an iterative construction method based on convolution to obtain a class of continuous vector refinable functions with arbitrary smoothness.

## 2.1 Piecewise continuous construction from Bernstein polynomials

In this section, we shall extend the definition (1.3.12) of $\mathbf{\Phi}$, as given in Example 1.3.1, to construct a piecewise continuous vector function $\mathbf{\Phi}^{[\nu]}$ of arbitrary length $\nu \in \mathbb{N}$, and with all component functions supported on $[0, 1]$.

For any non-negative integer $n$, the polynomial sequence $\{B_{n,k} : k = 0, \cdots, n\} \subset \pi_n$ defined by

$$B_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, \cdots, n, \tag{2.1.1}$$

are called the Bernstein polynomials of degree $n$.

Observe in particular from the definition (2.1.1) that the sequence $\{B_{n,k} : k = 0, \cdots, n\}$ satisfies the symmetry property

$$B_{n,n-k}(1-x) = B_{n,k}(x), \quad k = 0, \cdots, n. \tag{2.1.2}$$

For any $\nu \in \mathbb{N}$, let the vector function

$$\mathbf{\Phi} = \mathbf{\Phi}^{[\nu]} = \left(\phi_1^{[\nu]}, \cdots, \phi_\nu^{[\nu]}\right)^T : \mathbb{R} \to \mathbb{R}^\nu \tag{2.1.3}$$

be defined by

$$\phi_j^{[\nu]}(x) := \left\{ \begin{array}{l} B_{\nu-1,j-1}(x), \quad x \in [0,1); \\ \\ 0, \quad x \in \mathbb{R} \setminus [0,1), \end{array} \right\} \quad j = 1, \cdots, \nu, \tag{2.1.4}$$

where, from the definition (2.1.1), we have

$$B_{\nu-1,j-1}(x) := \binom{\nu-1}{j-1} x^{j-1} (1-x)^{\nu-j}, \quad j = 1, \cdots, \nu, \tag{2.1.5}$$

that is, $\{B_{\nu-1,j-1} : j = 1, \cdots, \nu\}$ are the Bernstein polynomials of degree $\nu - 1$. Note from (2.1.3)-(2.1.5) that the case $\nu = 1$ yields $\mathbf{\Phi}_1^{[1]} = \phi_1^{[1]} = N_0 = \chi_{[1,0)}$, the (scalar) box function (1.3.3), as drawn in Fig. 1.1(a), whereas the case $\nu = 2$ yields $\mathbf{\Phi}^{[2]} = \mathbf{\Phi}$, as defined in (1.3.12) of Example 1.3.1, and drawn in Fig. 1.2.

In order to establish the refinability of the vector function $\mathbf{\Phi} = \mathbf{\Phi}^{[\nu]}$, we shall rely on the following further properties of the Bernstein polynomials.

**Theorem 2.1.1** *For any non-negative integer $n$, the Bernstein polynomials $\{B_{n,k} : k = 0, \cdots, n\}$ of degree $n$, as defined in (2.1.1), satisfy the identities*

$$B_{n,k}(x) = \sum_{j=k}^{n} \frac{1}{2^j} \binom{j}{k} B_{n,j}(2x), \quad k = 0, \cdots, n, \tag{2.1.6}$$

*and*

$$B_{n,k}(x) = \sum_{j=0}^{k} \frac{1}{2^{n-j}} \binom{n-j}{n-k} B_{n,j}(2x-1), \quad k = 0, \cdots, n. \tag{2.1.7}$$

**Proof.** First, to prove (2.1.6), we apply the definition (2.1.1) to deduce that, for $k \in \{0, \cdots, n\}$ and any $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_{j=k}^{n} \frac{1}{2^j} \binom{j}{k} B_{n,j}(2x) &= \sum_{j=k}^{n} \frac{1}{2^j} \binom{j}{k} \binom{n}{j} (2x)^j (1-2x)^{n-j} \\
&= \sum_{j=k}^{n} \frac{j!}{k!(j-k)!} \frac{n!}{j!(n-j)!} x^j (1-2x)^{n-j} \\
&= \frac{n!}{k!} \sum_{j=k}^{n} \frac{x^j (1-2x)^{n-j}}{(j-k)!(n-j)!} \\
&= \frac{n!}{k!} \sum_{j=0}^{n-k} \frac{x^{j+k} (1-2x)^{n-k-j}}{j!(n-k-j)!} \\
&= \frac{n!}{k!} \frac{x^k}{(n-k)!} \sum_{j=0}^{n-k} \frac{(n-k)!}{j!(n-k-j)!} x^j (1-2x)^{n-k-j},
\end{aligned}$$

which implies that

$$\sum_{j=k}^{n} \frac{1}{2^j}\binom{j}{k} B_{n,j}(2x) = \binom{n}{k} x^k \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (1-2x)^{n-k-j}$$
$$= \binom{n}{k} x^k \big(x + (1-2x)\big)^{n-k} = \binom{n}{k} x^k (1-x)^{n-k} = B_{n,k}(x),$$

and it follows that the identity (2.1.6) holds.

To prove the identity (2.1.7), we first observe that (2.1.7) has the equivalent formulation

$$B_{n,k}\Big(x + \frac{1}{2}\Big) = \sum_{j=0}^{k} \frac{1}{2^{n-j}} \binom{n-j}{n-k} B_{n,j}(2x), \tag{2.1.8}$$

which, from the symmetry property (2.1.2) of the Bernstein polynomials, holds if and only if

$$B_{n,n-k}\Big(-x + \frac{1}{2}\Big) = \sum_{j=0}^{k} \frac{1}{2^{n-j}} \binom{n-j}{n-k} B_{n,j}(2x). \tag{2.1.9}$$

An application of (2.1.6) and (2.1.2) yields, for any $k \in \{0, \cdots, n\}$ and $x \in \mathbb{R}$,

$$B_{n,n-k}\Big(-x + \frac{1}{2}\Big) = \sum_{j=n-k}^{n} \frac{1}{2^j} \binom{j}{n-k} B_{n,j}(1-2x) = \sum_{j=0}^{k} \frac{1}{2^{n-j}} \binom{n-j}{n-k} B_{n,n-j}(1-2x)$$
$$= \sum_{j=0}^{k} \frac{1}{2^{n-j}} \binom{n-j}{n-k} B_{n,j}(2x),$$

which proves (2.1.9), and therefore also (2.1.7). ■

The following result, which extends Example 1.3.1, can now be established.

**Theorem 2.1.2** *For any $\nu \in \mathbb{N}$, the vector function $\mathbf{\Phi} = \mathbf{\Phi}^{[\nu]} : \mathbb{R} \to \mathbb{R}^\nu$, as defined in* (2.1.3)-(2.1.5)*, is refinable, with*

$$\mathbf{\Phi}^{[\nu]}(x) = \sum_k P^{[\nu]}(k)\, \mathbf{\Phi}^{[\nu]}(2x - k), \tag{2.1.10}$$

*where the matrix refinement sequence $\left\{P^{[\nu]}(k)\right\}$ is given by*

$$P^{[\nu]}(k) = \left[a_{ij}^{[\nu]}(k)\right]_{i,j=1,\cdots,\nu}, \quad k \in \mathbb{Z}, \tag{2.1.11}$$

*with*

$$a_{ij}^{[\nu]}(k) := \left\{ \begin{array}{l} \dfrac{1}{2^{j-1}}\dbinom{j-1}{i-1}, \quad k = 0; \\[2ex] \dfrac{1}{2^{\nu-j}}\dbinom{\nu-j}{\nu-i}, \quad k = 1; \\[2ex] 0, \quad k \in \mathbb{Z} \setminus \{0,1\}, \end{array} \right\}, \quad i,j = 1, \cdots, \nu. \tag{2.1.12}$$

**Proof.** First, observe from (2.1.3) and (2.1.11) that the identity (2.1.10) can be written as

$$\begin{bmatrix} \phi_1^{[\nu]}(x) \\ \phi_2^{[\nu]}(x) \\ \vdots \\ \phi_\nu^{[\nu]}(x) \end{bmatrix} = \sum_k \begin{bmatrix} a_{11}^{[\nu]}(k) & a_{12}^{[\nu]}(k) & \cdots & a_{1\nu}^{[\nu]}(k) \\ a_{21}^{[\nu]}(k) & a_{22}^{[\nu]}(k) & \cdots & a_{2\nu}^{[\nu]}(k) \\ \vdots & \vdots & & \vdots \\ a_{\nu 1}^{[\nu]}(k) & a_{\nu 2}^{[\nu]}(k) & \cdots & a_{\nu\nu}^{[\nu]}(k) \end{bmatrix} \begin{bmatrix} \phi_1^{[\nu]}(2x-k) \\ \phi_2^{[\nu]}(2x-k) \\ \vdots \\ \phi_\nu^{[\nu]}(2x-k) \end{bmatrix}, \tag{2.1.13}$$

or equivalently,

$$\phi_i^{[\nu]}(x) = \sum_{k=0}^{1} \sum_{j=1}^{\nu} a_{ij}^{[\nu]}(k) \phi_j^{[\nu]}(2x-k), \quad i = 1, \cdots, \nu. \tag{2.1.14}$$

To prove the identity (2.1.14), we fix $i \in \{1, \cdots, \nu\}$, and first let $x \in [0, 1/2)$. Since then $2x - 1 < 0$, and thus, from (2.1.4), $\phi_j^{[\nu]}(2x-1) = 0$, $j = 1, \cdots, \nu$, it follows that the identity (2.1.14) is equivalent to

$$\phi_i^{[\nu]}(x) = \sum_{j=1}^{\nu} a_{ij}^{[\nu]}(0) \phi_j^{[\nu]}(2x). \tag{2.1.15}$$

Also $2x \in [0, 1)$, so that, from (2.1.15), (2.1.4) and (2.1.12), we see that (2.1.15) may be written as

$$B_{\nu-1,i-1}(x) = \sum_{j=i}^{\nu} \frac{1}{2^{j-1}} \binom{j-1}{i-1} B_{\nu-1,j-1}(2x) = \sum_{j=i-1}^{\nu-1} \frac{1}{2^j} \binom{j}{i-1} B_{\nu-1,j}(2x), \quad i = 1, \cdots, \nu. \tag{2.1.16}$$

For any $i \in \{1, \cdots, \nu\}$, by applying the identity (2.1.6) in Theorem 2.1.1 with $n = \nu - 1$ and $k = i - 1$, so that $k \in \{0, \cdots, \nu - 1\}$, we deduce that

$$\sum_{j=i-1}^{\nu-1} \frac{1}{2^j} \binom{j}{i-1} B_{\nu-1,j}(2x) = B_{\nu-1,i-1}(x),$$

which proves (2.1.16), and therefore also (2.1.14), for $x \in [0, 1/2)$.

Next, for any $x \in [1/2, 1)$, i.e. $2x \in [1, 2)$, from (2.1.4), $\phi_j^{[\nu]}(2x) = 0$, $j = 1, \cdots, \nu$, according to which (2.1.14) is equivalent to the identity

$$\phi_i^{[\nu]}(x) = \sum_{j=1}^{\nu} a_{ij}^{[\nu]}(1) \phi_j^{[\nu]}(2x-1). \tag{2.1.17}$$

Also $2x - 1 \in [0, 1)$, so that we may use (2.1.4) and (2.1.12) to deduce that the identity (2.1.17) may be written as

$$B_{\nu-1,i-1}(x) = \sum_{j=1}^{i} \frac{1}{2^{\nu-j}} \binom{\nu-j}{\nu-i} B_{\nu-1,j-1}(2x-1). \tag{2.1.18}$$

For any $i \in \{1, \cdots, \nu\}$, by applying the identity (2.1.7) in Theorem 2.1.1, with $n = \nu - 1$ and $k = i - 1$, so that $k \in \{0, \cdots, \nu - 1\}$, we obtain

$$\sum_{j=1}^{i} \frac{1}{2^{\nu-j}} \binom{\nu-j}{\nu-i} B_{\nu-1,j-1}(2x-1) = \sum_{j=0}^{i-1} \frac{1}{2^{\nu-1-j}} \binom{\nu-1-j}{\nu-1-(i-1)} B_{\nu-1,j}(2x-1)$$
$$= B_{\nu-1,i-1}(x),$$

which proves (2.1.18), and therefore also (2.1.14) for $x \in [1/2, 1)$.

Finally, for $x \in \mathbb{R} \setminus [0, 1)$, it follows from (2.1.4) and (2.1.11), (2.1.12) that the identity (2.1.14) is satisfied with both sides equal to zero, and thereby completing our proof. ■

**Example 2.1.1** Following the construction in (2.1.3)-(2.1.5), the refinable vector function $\mathbf{\Phi}^{[3]}$ is given by

$$\mathbf{\Phi}^{[3]}(x) = \begin{bmatrix} \phi_1^{[3]}(x) \\ \phi_2^{[3]}(x) \\ \phi_3^{[3]}(x) \end{bmatrix} := \begin{bmatrix} B_{2,0}(x)\chi_{[0,1)}(x) \\ B_{2,1}(x)\chi_{[0,1)}(x) \\ B_{2,2}(x)\chi_{[0,1)}(x) \end{bmatrix} = \begin{bmatrix} (x^2-2x+1)\chi_{[0,1)}(x) \\ (-2x^2+2x)\chi_{[0,1)}(x) \\ x^2\chi_{[0,1)}(x) \end{bmatrix}, \tag{2.1.19}$$

and where its graph is given in Fig. 2.1.

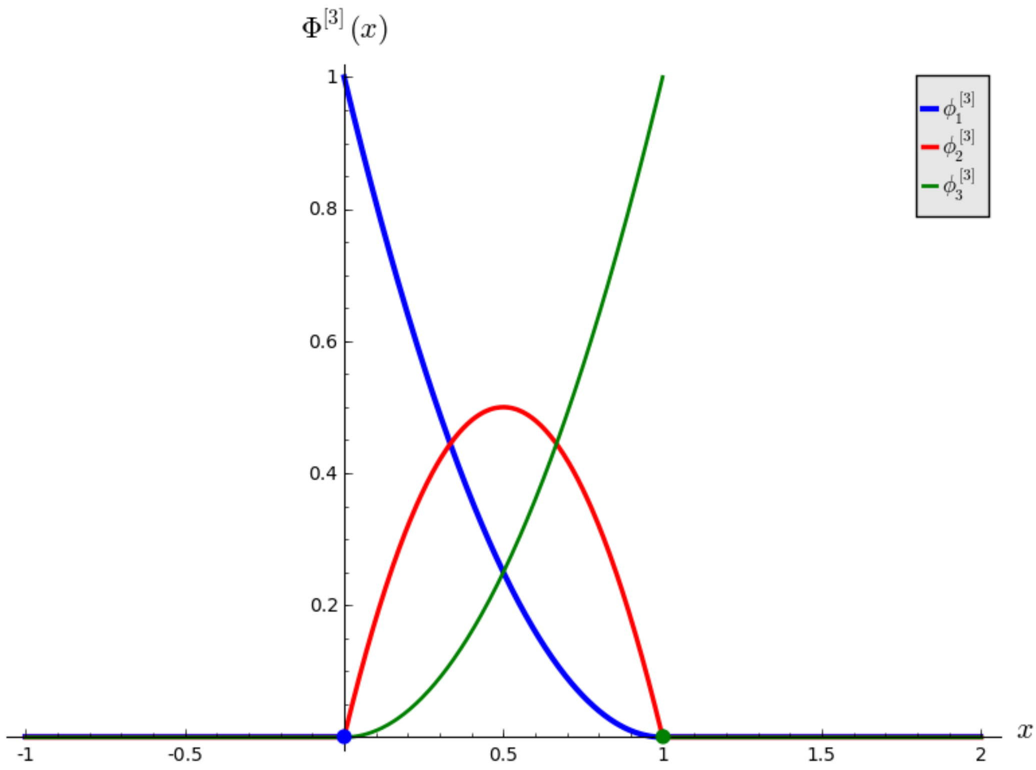



**Figure 2.1:** *Graph of* $\mathbf{\Phi}^{[3]} = \left(\phi_1^{[3]}, \phi_2^{[3]}, \phi_3^{[3]}\right)^T$, *as in* (2.1.19)

Also, by applying (2.1.11) and (2.1.12), we obtain the corresponding matrix refinement sequence

$$P^{[3]}(0) = \begin{bmatrix} 1 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/4 \end{bmatrix}; \quad P^{[3]}(1) = \begin{bmatrix} 1/4 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1 \end{bmatrix}; \quad P^{[3]}(k) = O, \quad k \in \mathbb{Z} \setminus \{0,1\}. \tag{2.1.20}$$

■

## 2.2 Smoothness from iterated vector convolution

According to Theorem 2.1.2, the vector function $\mathbf{\Phi} = \mathbf{\Phi}^{[\nu]}$ is refinable, with matrix refinement sequence $\{P(k)\} = \{P^{[\nu]}(k)\}$. However, we see from the definition (2.1.3)-(2.1.5) of $\mathbf{\Phi}^{[\nu]}$ that the two outer component functions $\phi_1^{[\nu]}$ and $\phi_\nu^{[\nu]}$ possess jump discontinuities at, respectively, 0 and 1, according to which $\mathbf{\Phi}^{[\nu]} \notin \mathbf{C}(\mathbb{R})$. In this section, we introduce an iterative convolution technique analogous to the scalar case in (1.3.4), to iteratively obtain a class of arbitrarily smooth refinable vector splines.

To this end, we first define the vector function $\mathcal{X}_{[0,1)}^{[\nu]}$ of length $\nu$ by

$$\mathcal{X}_{[0,1)}^{[\nu]} := (\chi_{[0,1)}, \cdots, \chi_{[0,1)})^T : \mathbb{R} \to \mathbb{R}^\nu, \tag{2.2.1}$$

where $\chi_{[0,1)}$ is the characteristic function on $[0,1)$, as previously applied in (1.3.4).

Then $\mathcal{X}_{[0,1)}^{[\nu]}$ is a refinable vector function with matrix refinement sequence $\{P(k)\}$ given by

$$P(k) = \begin{bmatrix} p_0(k) & 0 & \cdots & 0 \\ 0 & p_0(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_0(k) \end{bmatrix}, \tag{2.2.2}$$

with the sequence $\{p_0(k)\}$ given as in (1.3.9).

We proceed to introduce an iterative procedure to generate from the starting vector spline $\mathbf{\Phi}^{[\nu]}$ of Theorem 2.1.2, a sequence $\left\{\mathbf{\Phi}_k^{[\nu]} : k = 0, 1, \cdots\right\}$, with $\mathbf{\Phi}_0^{[\nu]} := \mathbf{\Phi}^{[\nu]}$, of refinable vector splines with increasing order of smoothness.

For any $\nu \in \mathbb{N}$ and compactly supported vector functions $\mathbf{f} = (f_1, \cdots, f_\nu)^T$, $\mathbf{g} = (g_1, \cdots, g_\nu)^T$, we define the convolution $\mathbf{h} = \mathbf{f} * \mathbf{g}$ as the vector function $\mathbf{h} = (h_1, \cdots, h_\nu)^T$ such that

$$h_k(x) := \int_{-\infty}^{\infty} f_k(x-t) g_k(t)\ dt, \quad k = 1, \cdots, \nu, \tag{2.2.3}$$

under the assumption also that the integrals on the right hand side of (2.2.3) exist.

Now let the vector function $\mathbf{\Phi}_0^{[\nu]} := \mathbf{\Phi}^{[\nu]}$ be defined as in (2.1.3)-(2.1.5), that is,

$$\mathbf{\Phi}_0^{[\nu]} = \left(\phi_{0,1}^{[\nu]}, \cdots, \phi_{0,\nu}^{[\nu]}\right)^T : \mathbb{R} \to \mathbb{R}^\nu, \tag{2.2.4}$$

with

$$\phi_{0,j}^{[\nu]}(x) := \begin{Bmatrix} B_{\nu-1,j-1}(x), \quad x \in [0,1); \\ 0, \quad x \in \mathbb{R} \setminus [0,1), \end{Bmatrix} \qquad j = 1, \cdots, \nu, \tag{2.2.5}$$

and where the Bernstein polynomials $\{B_{\nu-1,j} : j = 1, \cdots, \nu\}$ of degree $\nu - 1$ are given as in (2.1.5). The vector function sequence $\left\{\mathbf{\Phi}_m^{[\nu]} = \left(\phi_{m,1}^{[\nu]}, \cdots, \phi_{m,\nu}^{[\nu]}\right)^T : m = 1, 2, \cdots\right\}$ is now defined iteratively by means of the convolution

$$\mathbf{\Phi}_{m+1}^{[\nu]} := \mathbf{\Phi}_m^{[\nu]} * \mathcal{X}_{[0,1)}^{[\nu]}, \quad m = 0, 1, \cdots, \tag{2.2.6}$$

with the characteristic vector function $\mathcal{X}_{[0,1)}^{[\nu]} : \mathbb{R} \to \mathbb{R}^\nu$ defined as in (2.2.1), according to which (2.2.6) has the equivalent formulation

$$\mathbf{\Phi}_{m+1}^{[\nu]}(x) := \int_0^1 \mathbf{\Phi}_m^{[\nu]}(x-t)\ dt = \int_{x-1}^{x} \mathbf{\Phi}_m^{[\nu]}(t)\ dt, \quad m = 0, 1, \cdots, \tag{2.2.7}$$

with the definition

$$\int_a^b (f_1(t), \cdots, f_\nu(t))^T\ dt := \left(\int_a^b f_1(t)dt, \cdots, \int_a^b f_\nu(t)dt\right)^T,$$

for any interval $[a,b] \subset \mathbb{R}$.

We proceed to show that the convolution process (2.2.6), (2.2.7) preserves refinability, as follows.

**Theorem 2.2.1** *For any $\nu \in \mathbb{N}$, let $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T : \mathbb{R} \to \mathbb{R}^\nu$ be a refinable vector function with matrix refinement sequence $\{P(k)\}$, and with corresponding matrix refinement symbol $\mathcal{P}$ as defined in* (1.3.2). *Suppose also that every component function of $\mathbf{\Phi}$ is integrable on $\mathbb{R}$. Then:*

(a) *The convolution*

$$\widetilde{\mathbf{\Phi}}(x) := \big(\mathbf{\Phi} * \mathcal{X}_{[0,1)}^{[\nu]}\big)(x) = \int_0^1 \mathbf{\Phi}(x-t)\,dt = \int_{x-1}^{x} \mathbf{\Phi}(t)\ dt \tag{2.2.8}$$

*is also refinable, with matrix refinement sequence* $\{\widetilde{P}(k)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$ *given by*

$$\widetilde{P}(k) = \frac{1}{2}\big(P(k) + P(k-1)\big), \quad k \in \mathbb{Z}, \tag{2.2.9}$$

*and where the corresponding matrix refinement symbol* $\widetilde{\mathcal{P}}$ *is given by*

$$\widetilde{\mathcal{P}}(z) := \frac{1}{2}\sum_k \widetilde{P}(k)z^k = \Big(\frac{1+z}{2}\Big)\mathcal{P}(z). \tag{2.2.10}$$

(b) *If, moreover,* $\mathbf{\Phi}$ *is a piecewise constant function with respect to the integer partition* $\mathbb{Z}$ *of* $\mathbb{R}$*, then* $\widetilde{\mathbf{\Phi}} \in \mathbf{C}_0(\mathbb{R})$*, whereas if* $\mathbf{\Phi} \in \mathbf{C}_0^k(\mathbb{R})$ *for some non-negative integer* $k$*, then* $\widetilde{\mathbf{\Phi}} \in \mathbf{C}_0^{k+1}(\mathbb{R})$.

**Proof.**

(a) By applying the refinability of $\mathbf{\Phi}$, it follows from (2.2.8) that, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\widetilde{\mathbf{\Phi}}(x) &= \int_0^1 \mathbf{\Phi}(x-t)\,dt = \int_0^1 \sum_k P(k)\mathbf{\Phi}(2x-2t-k)\,dt \\
&= \sum_k P(k)\int_0^{1/2} \mathbf{\Phi}(2x-2t-k)\,dt + \sum_k P(k)\int_{1/2}^1 \mathbf{\Phi}(2x-2t-k)\,dt \\
&= \sum_k \frac{1}{2}P(k)\int_0^1 \mathbf{\Phi}(2x-t-k)\,dt + \sum_k \frac{1}{2}P(k)\int_1^2 \mathbf{\Phi}(2x-t-k)\,dt \\
&= \sum_k \frac{1}{2}P(k)\int_0^1 \mathbf{\Phi}(2x-t-k)\,dt + \sum_k \frac{1}{2}P(k)\int_0^1 \mathbf{\Phi}(2x-t-1-k)\,dt \\
&= \sum_k \frac{1}{2}P(k)\int_0^1 \mathbf{\Phi}(2x-t-k)\,dt + \sum_k \frac{1}{2}P(k-1)\int_0^1 \mathbf{\Phi}(2x-t-k)\,dt \\
&= \sum_k \frac{1}{2}\big(P(k)+P(k-1)\big)\int_0^1 \mathbf{\Phi}(2x-k-t)\,dt = \sum_k \frac{1}{2}\big(P(k)+P(k-1)\big)\widetilde{\mathbf{\Phi}}(2x-k),
\end{aligned}$$

which proves that $\widetilde{\mathbf{\Phi}}$ is a refinable vector function with matrix refinement sequence $\{\widetilde{P}(k)\}$ given by (2.2.9). Also, according to (2.2.9), we have, for any $z \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned}
\widetilde{\mathcal{P}}(z) = \frac{1}{2}\ \sum_k \frac{1}{2}\big(P(k)+P(k-1)\big)z^k &= \frac{1}{4}\sum_k P(k)z^k + \frac{1}{4}z\ \sum_k P(k-1)z^{k-1} \\
&= \frac{1}{4}\sum_k P(k)z^k + \frac{1}{4}z\ \sum_k P(k)z^k \\
&= \Big(\frac{1+z}{2}\Big)\Big(\frac{1}{2}\sum_k P(k)z^k\Big) = \Big(\frac{1+z}{2}\Big)\mathcal{P}(z),
\end{aligned}$$

and thereby proving (2.2.10).

(b) This property is an immediate consequence of the definition (2.2.8), together with, for $n \in \mathbb{N}$, the Fundamental Theorem of Calculus.

■

The following result is now an immediate consequence of the recursive formulation (2.2.4)-(2.2.7), together with Theorems 2.1.2 and 2.2.1.

**Theorem 2.2.2** *For any integers $\nu \in \mathbb{N}$ and $m \in \{0, 1, \cdots\}$, the vector function $\mathbf{\Phi}_m^{[\nu]} = \left(\phi_{m,1}^{[\nu]}, \cdots, \phi_{m,\nu}^{[\nu]}\right)^T$, as defined recursively by means of (2.2.4)-(2.2.7), is refinable, with corresponding matrix Laurent polynomial refinement symbol*

$$\mathcal{P}_m^{[\nu]}(z) = \left(\frac{1+z}{2}\right)^m \mathcal{P}_0^{[\nu]}(z), \tag{2.2.11}$$

*where*

$$\mathcal{P}_0^{[\nu]}(z) := \frac{1}{2}\sum_k P_0^{[\nu]}(k) z^k, \tag{2.2.12}$$

*with*

$$\left[P_0^{[\nu]}(k)\right]_{ij} := \left\{\begin{array}{l} \frac{1}{2^{j-1}}\binom{j-1}{i-1}, \quad k=0; \\ \frac{1}{2^{\nu-j}}\binom{\nu-j}{\nu-1}, \quad k=1; \\ 0, \quad k \in \mathbb{Z}\setminus\{0,1\}, \end{array}\right\}, \quad i,j = 1,\cdots,\nu. \tag{2.2.13}$$

*Moreover, $\mathbf{\Phi}_m^{[\nu]}$ is a vector spline, with*

$$\phi_{m,k}^{[\nu]}\Big|_{[j,j+1)} \in \pi_{\nu+m-1}, \quad j \in \mathbb{Z}, \quad k = 1,\cdots,\nu, \tag{2.2.14}$$

*and where*

$$\mathbf{\Phi}_m^{[\nu]} \in \mathbf{C}^{m-1}(\mathbb{R}). \tag{2.2.15}$$

*Also, $\mathbf{\Phi}_m^{[\nu]}$ is compactly supported on $\mathbb{R}$, with*

$$\text{supp } \phi_{m,k}^{[\nu]} = [0, m+1], \quad k = 1,\cdots,\nu. \tag{2.2.16}$$

**Remark 2.2.1** Observe that the case $\nu = 1$ of Theorem 2.2.2 corresponds precisely with the cardinal B-spline setting of (1.3.3), (1.3.4), (1.3.11), that is, $\mathbf{\Phi}_{m,1}^{[1]} = N_m$, the cardinal B-spline of degree $m$.

The vector spline $\mathbf{\Phi}_m^{[\nu]}$ of Theorem 2.2.2 has the following further properties.

**Theorem 2.2.3** *For any integer $\nu \in \mathbb{N}$ and $m \in \{0, 1, \cdots\}$, the refinable vector spline $\mathbf{\Phi}_m^{[\nu]} = \left(\phi_{m,1}^{[\nu]}, \cdots, \phi_{m,\nu}^{[\nu]}\right)^T$ of Theorem 2.2.2 satisfies the following properties:*

$$\text{(a)} \qquad \phi_{m,k}^{[\nu]}(x) > 0, \quad x \in (0, m+1), \quad k = 1, \cdots, \nu; \tag{2.2.17}$$

$$\text{(b)} \qquad \phi_{m,k}^{[\nu]}(x) = \phi_{m,\nu+1-k}^{[\nu]}(m+1-x), \quad x \in \mathbb{R}, \quad k = 1, \cdots, \nu; \tag{2.2.18}$$

$$\text{(c)} \qquad \sum_{k=1}^{\nu} \sum_{j} \phi_{m,k}^{[\nu]}(x-j) = 1, \quad x \in \mathbb{R}. \tag{2.2.19}$$

**Proof.**

(a) First, observe from (2.2.6) and (2.1.5) that (2.2.17) holds for $m = 0$. An inductive proof based on the recursive formulation (2.2.7) then shows that (2.2.17) is satisfied for each $m \in \{0, 1, \cdots\}$.

(b) By applying (2.2.6) and (2.1.2), we find that (2.2.18) holds for $m = 0$. Proceeding inductively, suppose next that (2.2.18) holds for a fixed non-negative integer $m$. It then follows from (2.2.7) that, for any $k \in \{1, \cdots, \nu\}$ and $x \in \mathbb{R}$,

$$\begin{aligned}
\phi_{m+1,\nu+1-k}^{[\nu]}(m+2-x) &= \int_0^1 \phi_{m,\nu+1-k}^{[\nu]}(m+2-x-t)\, dt \\
&= \int_0^1 \phi_{m,k}^{[\nu]}\big((m+1)-(m+2-x-t)\big)\, dt \\
&= \int_0^1 \phi_{m,k}^{[\nu]}\big(x-(1-t)\big)\, dt \\
&= \int_0^1 \phi_{m,k}^{[\nu]}(x-t)\, dt = \phi_{m+1,k}^{[\nu]}(x),
\end{aligned}$$

which advances the inductive hypothesis from $m$ to $m+1$, and thereby completing our proof of (2.2.18).

(c) For $x \in \mathbb{R}$, let $j_0$ denote the (unique) integer such that $x \in [j_0, j_0+1)$, and thus $x - j_0 \in [0, 1)$. It follows from (2.2.6) and (2.2.5) that

$$\begin{aligned}
\sum_{k=1}^{\nu} \sum_{j} \phi_{0,k}^{[\nu]}(x-j) &= \sum_{k=1}^{\nu} \phi_{0,k}^{[\nu]}(x-j_0) \\
&= \sum_{k=1}^{\nu} B_{\nu-1,k-1}(x-j_0) = \sum_{k=1}^{\nu} \binom{\nu-1}{k-1} (x-j_0)^{k-1}(1-x+j_0)^{\nu-k},
\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{k=1}^{\nu}\sum_{j}\phi_{0,k}^{[\nu]}(x-j) &= \sum_{k=0}^{\nu-1}\binom{\nu-1}{k}(x-j_0)^k(1-x+j_0)^{\nu-1-k}\\ &= \Big[(x-j_0)+(1-x+j_0)\Big]^{\nu-1} = 1^{\nu-1} = 1,\end{aligned}$$

according to which the property (2.2.19) holds for $m=0$. Suppose next (2.2.19) holds for a fixed non-negative integer $m$. It then follows from (2.2.7) and the inductive hypothesis that, for any $x \in \mathbb{R}$,

$$\begin{aligned}\sum_{k=1}^{\nu}\sum_{j}\phi_{m+1,k}^{[\nu]}(x-j) = \sum_{k=1}^{\nu}\sum_{j}\int_0^1 \phi_{m,k}^{[\nu]}(x-j-t)\,dt &= \int_0^1\left[\sum_{k=1}^{\nu}\sum_{j}\phi_{m,k}^{[\nu]}(x-t-j)\right]dt\\ &= \int_0^1 1\,dt = 1,\end{aligned}$$

which then completes our inductive proof of (2.2.19).

■

By applying the explicit formulations (2.2.4)-(2.2.7) and (2.2.11)-(2.2.13), as well as (2.2.9), we proceed to explicitly calculate the following examples of refinable vector splines $\mathbf{\Phi}_m^{[\nu]}$, as well as their matrix refinement sequences $\left\{P_m^{[\nu]}(k)\right\}$ of Theorem 2.2.2.

**Example 2.2.1** Application of (2.2.4)-(2.2.7) and Theorem 2.2.2 for $\nu = 2, 3,\ m = 1, 2$.

- $\nu = 2,\ m = 1$

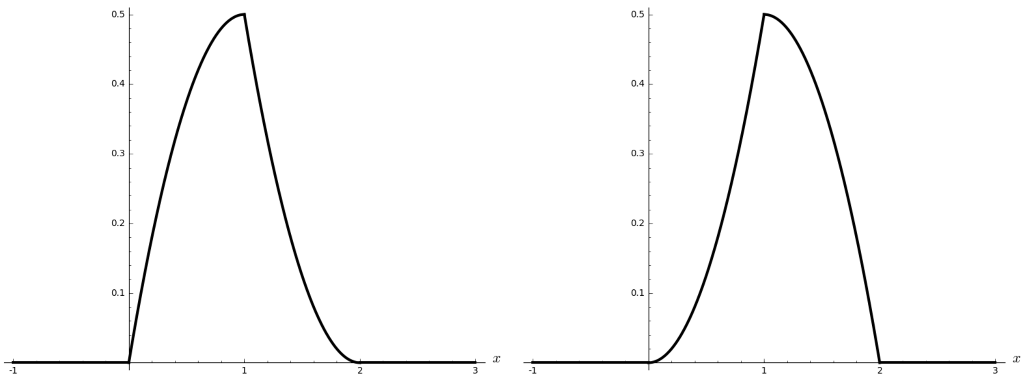
  The refinable vector spline $\mathbf{\Phi}_1^{[2]} = \left(\phi_{1,1}^{[2]}, \phi_{1,2}^{[2]}\right)^T$ is defined by

$$\begin{aligned}\mathbf{\Phi}_1^{[2]}(x) &:= \left(\mathbf{\Phi}_0^{[2]} * \mathcal{X}_{[0,1)}^{[2]}\right)(x)\\ &= \left\{\begin{array}{ll}\left(-\frac{1}{2}x^2+x,\ \frac{1}{2}x^2\right)^T, & x\in[0,1),\\ \left(\frac{1}{2}x^2-2x+2,\ -\frac{1}{2}x^2+x\right)^T, & x\in[1,2),\\ \mathbf{0}, & x\in\mathbb{R}\setminus[0,2).\end{array}\right\}\end{aligned} \tag{2.2.20}$$

  Also, its corresponding matrix refinement sequence $\left\{P_1^{[2]}(k)\right\} \in l_0^{2\times 2}(\mathbb{Z})$ is given, as follows.

$$\left.\begin{array}{l}P_1^{[2]}(0) = \frac{1}{4}\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix};\quad P_1^{[2]}(1) = \frac{1}{4}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix};\quad P_1^{[2]}(2) = \frac{1}{4}\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix};\\ P_1^{[2]}(k) = O,\quad k\in\mathbb{Z}\setminus\{0,1,2\}.\end{array}\right\} \tag{2.2.21}$$

Moreover, the graph of $\mathbf{\Phi}_1^{[2]}$ is shown in Fig. 2.2.




**(a)** $\phi_{1,1}^{[2]}$ **(b)** $\phi_{1,2}^{[2]}$

**Figure 2.2:** *Graph of* $\mathbf{\Phi}_1^{[2]} = \left(\phi_{1,1}^{[2]}, \phi_{1,2}^{[2]}\right)^T$

- $\nu = 2, \ m = 2$

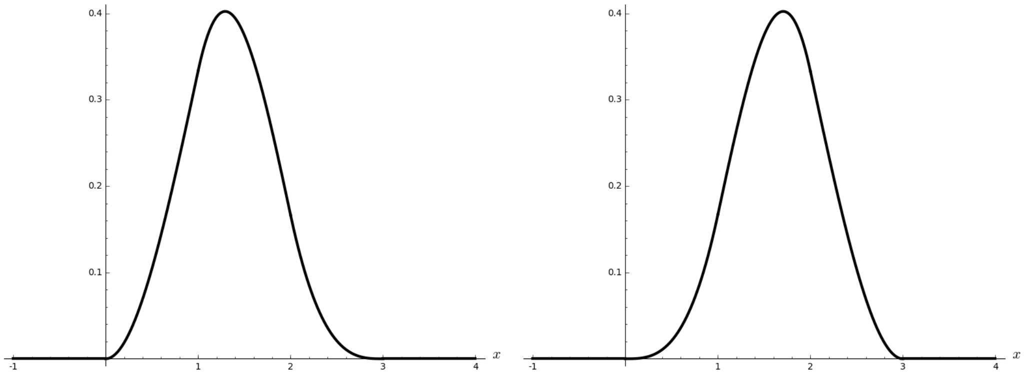
The refinable vector spline $\mathbf{\Phi}_2^{[2]} = \left(\phi_{2,1}^{[2]}, \phi_{2,2}^{[2]}\right)^T$ is given by

$$\mathbf{\Phi}_2^{[2]}(x) := \left(\mathbf{\Phi}_1^{[2]} * \mathcal{X}_{[0,1)}^{[2]}\right)(x)$$
$$= \left\{\begin{array}{ll} \left(-\frac{1}{6}x^3 + \frac{1}{2}x^2, \ \frac{1}{6}x^3\right)^T, & x \in [0,1), \\ \left(\frac{1}{3}x^3 - 2x^2 + \frac{7}{2}x - \frac{3}{2}, \ -\frac{1}{3}x^3 + x^2 - \frac{1}{2}x\right)^T, & x \in [1,2), \\ \left(-\frac{1}{6}x^3 + \frac{3}{2}x^2 - \frac{9}{2}x + \frac{9}{2}, \ \text{‘}\frac{1}{6}x^3 - x^2 + \frac{3}{2}x\right)^T, & x \in [2,3), \\ \quad \mathbf{0}, & x \in \mathbb{R} \setminus [0,3). \end{array}\right\}, \tag{2.2.22}$$

Its corresponding matrix refinement sequence $\left\{P_2^{[2]}(k)\right\} \in l_0^{2\times 2}(\mathbb{Z})$ is expressed, as follows.

$$\left.\begin{array}{l} P_2^{[2]}(0) = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}; \quad P_2^{[2]}(1) = \frac{1}{8}\begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}; \quad P_2^{[2]}(2) = \frac{1}{8}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}; \quad P_2^{[2]}(3) = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}; \\ P_2^{[2]}(k) = O, \quad k \in \mathbb{Z} \setminus \{0,1,2,3\}. \end{array}\right\} \tag{2.2.23}$$

The graph of $\mathbf{\Phi}_2^{[2]}$ is given in Fig. 2.3.




**(a)** $\phi_{2,1}^{[2]}$ **(b)** $\phi_{2,2}^{[2]}$

**Figure 2.3:** *Graph of* $\mathbf{\Phi}_2^{[2]} = \left(\phi_{2,1}^{[2]}, \phi_{2,2}^{[2]}\right)^T$

- $\nu = 3,\ m = 1$

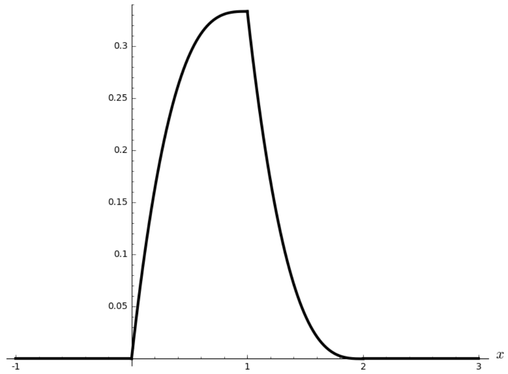
  The refinable vector spline $\mathbf{\Phi}_1^{[3]}$ is given by

$$\mathbf{\Phi}_1^{[3]}(x) = \left(\phi_{3,1}^{[3]}, \phi_{3,2}^{[3]}, \phi_{3,3}^{[3]}\right)^T := \left(\mathbf{\Phi}_0^{[3]} * \mathcal{X}_{[0,1)}^{[3]}\right)(x)$$
$$= \begin{bmatrix} (\frac{1}{3}x^3 - x^2 + x)\chi_{[0,1)}(x) + (-\frac{1}{3}x^3 + 2x^2 - 4x + \frac{8}{3})\chi_{[1,2)}(x) \\ (-\frac{2}{3}x^3 + x^2)\chi_{[0,1)}(x) + (\frac{2}{3}x^3 - 3x^2 + 4x - \frac{4}{3})\chi_{[1,2)}(x) \\ (\frac{1}{3}x^3)\chi_{[0,1)}(x) + (-\frac{1}{3}x^3 + x^2 - x + \frac{2}{3})\chi_{[1,2)}(x) \end{bmatrix} \tag{2.2.24}$$

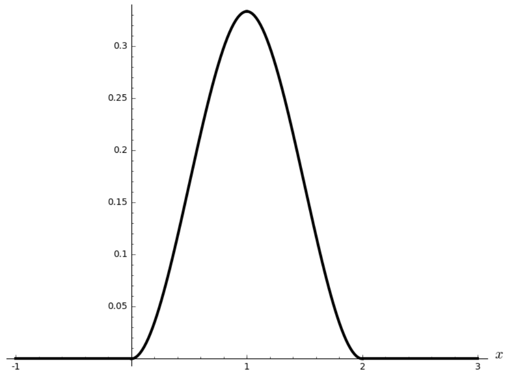
  Its corresponding matrix refinement sequence $\left\{P_1^{[3]}(k)\right\} \in l_0^{3\times 3}(\mathbb{Z})$ is expressed, as follows.

$$\left.\begin{aligned} &P_1^{[3]}(0) = \frac{1}{8}\begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_1^{[3]}(1) = \frac{1}{8}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}; \quad P_1^{[3]}(2) = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix}; \\ &P_1^{[3]}(k) = O, \quad k \in \mathbb{Z} \setminus \{0, 1, 2\}. \end{aligned}\right\} \tag{2.2.25}$$

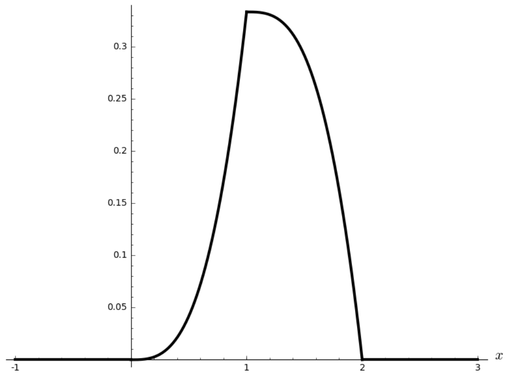
  The graph of $\mathbf{\Phi}_1^{[3]}$ is given in Fig. 2.4.




**(a)** $\phi_{1,1}^{[3]}$




**(b)** $\phi_{1,2}^{[3]}$




**(c)** $\phi_{1,3}^{[3]}$

**Figure 2.4:** *Graph of* $\mathbf{\Phi}_1^{[3]} = \left(\phi_{1,1}^{[3]}, \phi_{1,2}^{[3]}, \phi_{1,3}^{[3]}\right)^T$

- $\nu = 3,\ m = 2$

  The refinable vector spline $\mathbf{\Phi}_2^{[3]}$ is given by

$$\mathbf{\Phi}_2^{[3]}(x) = \left(\phi_{3,1}^{[3]}, \phi_{3,2}^{[3]}, \phi_{3,3}^{[3]}\right)^T := \left(\mathbf{\Phi}_1^{[3]} * \mathcal{X}_{[0,1)}^{[3]}\right)(x), \tag{2.2.26}$$

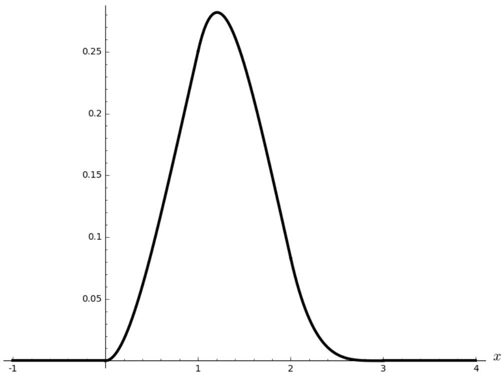
  where

$$\left\{\begin{aligned} \phi_{2,1}^{[3]}(x) &= (\tfrac{1}{12}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2)\chi_{[0,1)}(x) + (-\tfrac{1}{6}x^4 + \tfrac{4}{3}x^3 - 4x^2 + 5x - \tfrac{23}{12})\chi_{[1,2)}(x) \\ &\qquad + (\tfrac{1}{12}x^4 - x^3 + \tfrac{9}{2}x^2 - 9x + \tfrac{27}{4})\chi_{[2,3)}(x) \\ \phi_{2,2}^{[3]}(x) &= (-\tfrac{1}{6}x^4 + \tfrac{1}{3}x^3)\chi_{[0,1)}(x) + (\tfrac{1}{3}x^4 - 2x^3 + 4x^2 - 3x + \tfrac{5}{6})\chi_{[1,2)}(x) \\ &\qquad + (-\tfrac{1}{6}x^4 + \tfrac{5}{3}x^3 - 6x^2 + 9x - \tfrac{9}{2})\chi_{[2,3)}(x) \\ \phi_{2,3}^{[3]}(x) &= \tfrac{1}{12}x^4\chi_{[0,1)}(x) + (-\tfrac{1}{6}x^4 + \tfrac{2}{3}x^3 - x^2 + x - \tfrac{5}{12})\chi_{[1,2)}(x) \\ &\qquad + (\tfrac{1}{12}x^4 - \tfrac{2}{3}x^3 + 2x^2 - 3x + \tfrac{9}{4})\chi_{[2,3)}(x). \end{aligned}\right\} \tag{2.2.27}$$

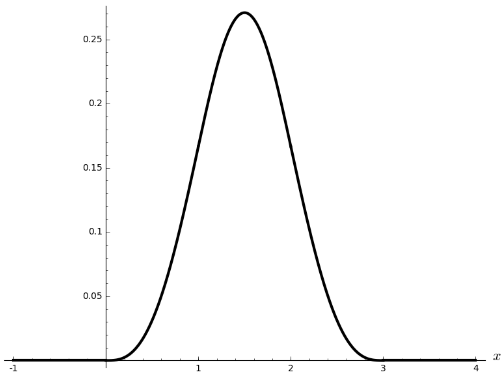
Its corresponding matrix refinement sequence $\left\{P_2^{[3]}(k)\right\} \in l_0^{3\times 3}(\mathbb{Z})$ is expressed, as follows.

$$\left.\begin{array}{l} P_2^{[3]}(0) = \frac{1}{16}\begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}; \quad P_2^{[3]}(1) = \frac{1}{16}\begin{bmatrix} 9 & 4 & 2 \\ 2 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}; \quad P_2^{[3]}(2) = \frac{1}{16}\begin{bmatrix} 6 & 2 & 1 \\ 4 & 6 & 2 \\ 2 & 4 & 9 \end{bmatrix}; \\ P_2^{[3]}(3) = \frac{1}{16}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix}; \quad P_2^{[3]}(k) = O, \quad k \in \mathbb{Z} \setminus \{0,1,2,3\}. \end{array}\right\} \tag{2.2.28}$$

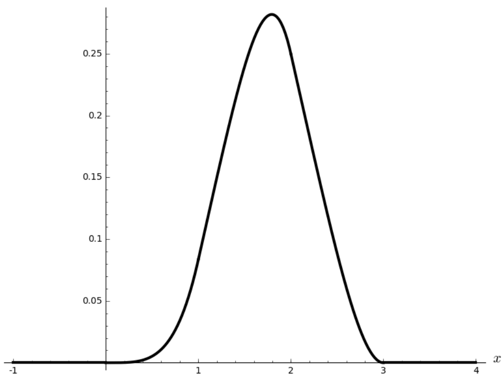
The graph of $\mathbf{\Phi}_2^{[3]}$ is given in Fig. 2.5.




**(a)** $\phi_{2,1}^{[3]}$




**(b)** $\phi_{2,2}^{[3]}$




**(c)** $\phi_{2,3}^{[3]}$

**Figure 2.5:** *Graph of* $\mathbf{\Phi}_2^{[3]} = \left(\phi_{2,1}^{[3]}, \phi_{2,2}^{[3]}, \phi_{2,3}^{[3]}\right)^T$

$\blacksquare$

## 2.3 Linear independence and stability analysis from Fourier transforms

We proceed in this section to investigate the integer-shift linear independence and stability, as described in, respectively, Sections 1.4 and 1.5, of the vector spline $\mathbf{\Phi}_m^{[\nu]}$ constructed in Section 2.2. Note from Example 1.4.1 that the vector spline $\mathbf{\Phi}_0^{[2]}$ has already been shown to possess matrix linearly independent integer shifts on $\mathbb{R}$. As previously observed in Section 1.5, an application of Theorem 1.5.2 then shows that $\mathbf{\Phi}_0^{[2]}$ also possesses $l^2$-stable integer shifts on $\mathbb{R}$. We next show that such integer-shift linear independence and $l^2$-stability are achieved by $\mathbf{\Phi}_0^{[\nu]}$ for all $\nu \in \mathbb{N}$.

**Theorem 2.3.1** *For any integer $\nu \in \mathbb{N}$, the vector function $\mathbf{\Phi}_0^{[\nu]} = \left(\phi_{0,1}^{[\nu]}, \cdots, \phi_{0,\nu}^{[\nu]}\right)^T$, as defined by* (2.2.4), (2.2.5), *possesses matrix linearly independent integer shifts, as well as $l^2$-stable integer shifts, on* $\mathbb{R}$.

**Proof.** Let $\{a_1(k)\}, \cdots, \{a_\nu(k)\}$ be sequences in $l(\mathbb{Z})$ satisfying the identity

$$\sum_{j=1}^{\nu} \sum_{k} a_j(k) \phi_{0,j}^{[\nu]}(x-k) = 0, \quad x \in \mathbb{R}. \tag{2.3.1}$$

Let $l \in \mathbb{Z}$. It then follows from (2.2.5) and (2.3.1) that

$$\sum_{j=1}^{\nu} a_j(l) B_{\nu-1,j-1}(x-l) = 0, \quad x \in [l, l+1),$$

and thus

$$\sum_{j=0}^{\nu-1} a_{j+1}(l) B_{\nu-1,j}(x) = 0, \quad x \in \mathbb{R}. \tag{2.3.2}$$

Since, moreover, $\left\{B_{\nu-1,j} \ : \ j = 0, \cdots, \nu-1\right\}$ is the Bernstein basis for the polynomial space $\pi_{\nu-1}$, and therefore a linearly independent set, it follows from (2.3.2) that

$$a_j(l) = 0, \quad j = 1, \cdots, \nu. \tag{2.3.3}$$

Since also the integer $l \in \mathbb{Z}$ was chosen arbitrarily, we may deduce from (2.3.3) that

$$a_j(l) = 0, \quad l \in \mathbb{Z}; \ j = 1, \cdots, \nu. \tag{2.3.4}$$

Hence (2.3.1) implies (2.3.4), according to which $\mathbf{\Phi}_0^{[\nu]}$ possesses linearly independent integer shifts on $\mathbb{R}$.

The stability statement of the theorem is now an immediate consequence of Theorem 1.5.2. ∎

For the scalar case $\nu = 1$, where, as noted before in Remark 2.2.1, we have $\mathbf{\Phi}_{m,1}^{[1]} = N_m$, the cardinal B-spline of degree $m$, it is known (see e.g. [2] and [3]) that $N_m$ possesses linearly independent integer shifts on $\mathbb{R}$, as well as $l^2$-stable integer shifts on $\mathbb{R}$.

In order to investigate the shift linear independence and $l^2$-stability of $\mathbf{\Phi}_m^{[\nu]}$ for $\nu \geq 2$ and $m \in \mathbb{N}$, we now introduce the concept of Fourier transforms, as follows.

We shall denote by $L^1(\mathbb{R})$ the space of functions $f : \mathbb{R} \to \mathbb{C}$ such that the (Lebesgue) integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ is finite. Observe that $C_0(\mathbb{R}) \subset L^1(\mathbb{R})$.

For any compactly supported function $f \in L^1(\mathbb{R})$, the Fourier transform $\mathcal{F}f = \widehat{f} : \mathbb{C} \to \mathbb{C}$ of $f$ is defined by

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx, \quad \omega \in \mathbb{C}, \tag{2.3.5}$$

where $i$ is the imaginary unit. Observe in particular from (2.3.5) that

$$\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx. \tag{2.3.6}$$

We shall rely on the Fourier transform results in Theorems 2.3.2 -2.3.5 below.

**Theorem 2.3.2** *For any compactly supported function $f \in L^1(\mathbb{R})$, let*

$$g(x) := \left(f * \chi_{[0,1)}\right)(x) = \int_0^1 f(x-t)\, dt, \quad x \in \mathbb{R}. \tag{2.3.7}$$

*Then $g$ is a compactly supported function in $L^1(\mathbb{R})$, with Fourier transform $\widehat{g}$ given by*

$$\widehat{g}(\omega) = \begin{cases} \left(\dfrac{1-e^{-i\omega}}{i\omega}\right)\widehat{f}(\omega), & \omega \in \mathbb{C} \setminus \{0\}; \\ \widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx, & \omega = 0. \end{cases} \tag{2.3.8}$$

**Proof.** For $\omega \in \mathbb{C} \setminus \{0\}$, it follows from (2.3.5) and (2.3.7) that

$$\begin{aligned} \widehat{g}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\left[\int_0^1 f(x-t)\, dt\right] dx &= \int_0^1 \left[\int_{-\infty}^{\infty} e^{-i\omega x} f(x-t)\, dx\right] dt \\ &= \int_0^1 \left[\int_{-\infty}^{\infty} e^{-i\omega(x+t)} f(x)\, dx\right] dt \\ &= \int_0^1 e^{-i\omega t}\left[\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx\right] dt \\ &= \left[\int_0^1 e^{-i\omega t} dt\right]\left[\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx\right] \\ &= \left(\frac{1-e^{-i\omega}}{i\omega}\right)\widehat{f}(\omega), \end{aligned}$$

which proves the first line of (2.3.8).

For $\omega = 0$, we apply (2.3.6) and (2.3.7) to deduce that

$$\begin{aligned}\widehat{g}(0) = \int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty}\left[\int_0^1 f(x-t)\ dt\right]dx &= \int_0^1\left[\int_{-\infty}^{\infty} f(x-t)\ dx\right]dt\\ &= \int_0^1\left[\int_{-\infty}^{\infty} f(x)\ dx\right]dt\\ &= \left[\int_0^1 dt\right]\left[\int_{-\infty}^{\infty} f(x)\ dx\right] = \int_{-\infty}^{\infty} f(x)\ dx,\end{aligned}$$

which proves the second line of (2.3.8). ■

Our next two results on equivalent Fourier transform formulations of, respectively, matrix linear independence and $l^2$-stability are from [36, Sections 4 and 5].

**Theorem 2.3.3** *For any $\nu \in \mathbb{N}$, let $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T : \mathbb{R} \to \mathbb{R}^\nu$ denote a compactly supported vector function, with $\phi_k \in L^1(\mathbb{R})$, $k = 1, \cdots, \nu$. Then $\mathbf{\Phi}$ possesses matrix linearly independent integer shifts on $\mathbb{R}$, if and only if, for each $\omega \in \mathbb{C}$, the only coefficient sequence $\{c_1, \cdots, c_\nu\} \subset \mathbb{C}$ satisfying the condition*

$$\sum_{j=1}^{\nu} c_j \widehat{\phi}_j(\omega + 2\pi k) = 0, \quad k \in \mathbb{Z}, \tag{2.3.9}$$

*is the zero sequence, that is*

$$c_j = 0, \quad j = 1, \cdots, \nu. \tag{2.3.10}$$

**Theorem 2.3.4** *For any $\nu \in \mathbb{N}$, let $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T : \mathbb{R} \to \mathbb{R}^\nu$ denote a compactly supported vector function, with $\phi_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $k = 1, \cdots, \nu$. Then $\mathbf{\Phi}$ possesses $l^2$-stable integer shifts on $\mathbb{R}$ if and only if, for each $\omega \in \mathbb{R}$, the only coefficient sequence $\{c_1, \cdots, c_\nu\} \subset \mathbb{C}$ satisfying the condition* (2.3.9) *is the zero sequence as in* (2.3.10).

Note from Theorems 2.3.3 and 2.3.4 that linear independence implies stability, as previously formulated in Theorem 1.5.2. By applying Theorems 2.3.2-2.3.4, we next prove the following negative result.

**Theorem 2.3.5** *For any integer $\nu \geq 2$, let $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T$ denote a compactly supported vector function, with also $\phi_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $j = 1, \cdots, \nu$, and where, for at least two indexes $j$ in the set $\{1, \cdots, \nu\}$, it holds that*

$$\int_{-\infty}^{\infty} \phi_j(x)\ dx \neq 0. \tag{2.3.11}$$

*Then the vector function* $\mathbf{\Phi}^* = \left(\phi_1^*, \cdots, \phi_\nu^*\right)^T$ *defined by*

$$\phi_j^*(x) := \left(\phi_j * \chi_{[0,1)}\right)(x) = \int_0^1 \phi_j(x-t)\ dt, \quad j = 1, \cdots, \nu, \tag{2.3.12}$$

*possesses neither matrix linearly independent integer shifts, nor* $l^2$*-stable integer shifts, on* $\mathbb{R}$.

**Proof.** We shall show that there exists a coefficient sequence $\{c_1, \cdots, c_\nu\} \subset \mathbb{C}$, with $c_j \neq 0$ for at least one index $j \in \{1, \cdots, \nu\}$, and such that

$$\sum_{j=1}^{\nu} c_j \widehat{\phi}_j^*(2\pi k) = 0, \quad k \in \mathbb{Z}, \tag{2.3.13}$$

which, together with the case $\omega = 0$ of both Theorems 2.3.3 and 2.3.4, will then complete our proof.

To this end, we first apply (2.3.8) in Theorem 2.3.2 to deduce that

$$\widehat{\phi}_j^*(2\pi k) = \left\{\begin{matrix} 0, \quad k \in \mathbb{Z} \setminus \{0\}; \\ \int_{-\infty}^{\infty} \phi_j(x)dx, \quad k = 0, \end{matrix}\right\}, \quad j = 1, \cdots, \nu, \tag{2.3.14}$$

and thus

$$\sum_{j=1}^{\nu} c_j \widehat{\phi}_j^*(2\pi k) = \begin{cases} 0, \quad k \in \mathbb{Z} \setminus \{0\}; \\ \displaystyle\sum_{j=1}^{\nu} c_j \Big\{ \int_{-\infty}^{\infty} \phi_j(x)dx \Big\}, \quad k = 0, \end{cases} \tag{2.3.15}$$

from which it then follows that the condition (2.3.11) is equivalent to the single equation

$$\sum_{j=1}^{\nu} \Big\{ \int_{-\infty}^{\infty} \phi_j(x)dx \Big\} c_j = 0. \tag{2.3.16}$$

Since, moreover, we have $\nu \geq 2$, and the condition (2.3.11) is satisfied for at least two indexes $j$ in the set $\{1, \cdots, \nu\}$, we deduce from (2.3.16) that there exists a sequence $\{c_1, \cdots, c_\nu\} \subset \mathbb{C}$, with $c_j \neq 0$ for at least one index $j$ in the set $\{1, \cdots, \nu\}$, such that (2.3.13) is satisfied, and thereby completing our proof. ■

Now observe from Theorem 2.2.3(a) that, for any non-negative integer $m$, the refinable vector spline $\mathbf{\Phi}_m^{[\nu]} = \left(\phi_{m,1}^{[\nu]}, \cdots, \phi_{m,\nu}^{[\nu]}\right)^T$ of Theorem 2.2.2 satisfies the condition

$$\int_{-\infty}^{\infty} \phi_{m,j}^{[\nu]}(x)dx > 0, \quad j = 1, \cdots, \nu. \tag{2.3.17}$$

Hence we may apply Theorems 2.3.1 and 2.3.3- 2.3.5 to deduce the following result.

**Theorem 2.3.6** *For any integers $\nu \geq 2$ and $m \in \{0, 1, \cdots\}$, the refinable vector spline $\mathbf{\Phi}_m^{[\nu]}$ of Theorem 2.2.2 possesses matrix linearly independent integer shifts, as well as $l^2$-stable integer shifts, on $\mathbb{R}$, if and only if $m = 0$.*

The refinable vector spline $\mathbf{\Phi}_m^{[\nu]}$, although lacking the linear independence and stability properties, was shown in [30], for the case $\nu = 2$, to nevertheless have useful application possiblities in vector subdivision.

We shall proceed in Chapter 3 to construct smooth refinable vector splines which do possess the properties of linear independence and $l^2$-stability.

# Chapter 3

# SMOOTH REFINABILITY WITH LINEAR INDEPENDENCE

In this chapter, we construct a class of arbitrarily smooth refinable vector splines which also possess matrix linearly independent integer shifts, as well as $l^2$-stable integer shifts, on $\mathbb{R}$. Our work extends the refinable vector spline contruction of [31], in which one of the spline components was non-smooth.

## 3.1 Construction by means of truncated powers

For any integers $n \in \mathbb{N}$ and $k \in \{1, \cdots, \sigma_n\}$, where

$$\sigma_n := \max\{1, n-1\}, \tag{3.1.1}$$

the cardinal spline space $S_{n,k}(\mathbb{Z})$ of degree $n$ and deficiency $k$ is defined by

$$S_{n,k}(\mathbb{Z}) := \left\{ f \in C^{n-1-k}(\mathbb{R}) : \ f|_{[j,j+1)} \in \pi_n, \ j \in \mathbb{Z} \right\}, \tag{3.1.2}$$

where $S_{1,1}(\mathbb{Z})$ denotes the space of piecewise linear polynomials with respect to the integer partition $\mathbb{Z}$ of $\mathbb{R}$.

Our aim here is to construct a spline $G \in S_{n,k}(\mathbb{Z})$ such that $G$ is compactly supported, with

$$\text{supp } G = [0, n], \tag{3.1.3}$$

and with the only discontinuities in $G$ and its derivatives occurring in $G^{(n)}$ at all of the points $\{0, \cdots, n\}$ and in $G^{(n-k)}$ at 0, with one-sided derivatives

$$G^{(n-k)}(0_-) = 0; \quad G^{(n-k)}(0_+) = 1. \tag{3.1.4}$$

To this end we introduce, for any non-negative integer $m$, the truncated power function $(\cdot)_+^m : \mathbb{R} \to \mathbb{R}$, as defined by

$$x_+^m := \begin{cases} x^m, & x \geq 0; \\ 0, & x < 0, \end{cases} \tag{3.1.5}$$

where $0^0 := 1$, and according to which

$$(\cdot - j)_+^m \in C^{m-1}(\mathbb{R}), \quad j \in \mathbb{Z}. \tag{3.1.6}$$

It then follows from (3.1.5) and (3.1.6) that, for any real coefficient sequence $\{\alpha(j) : \ j = 0, \cdots, n\}$, the construction

$$G(x) := \frac{x_+^{n-k}}{(n-k)!} + \sum_{j=0}^{n} \alpha(j)(x-j)_+^n, \quad x \in \mathbb{R}, \tag{3.1.7}$$

yields a spline $G \in S_{n,k}(\mathbb{Z})$, with the only discontinuities in $G$ and its derivatives occurring in $G^{(n)}$ at all of the points $\{0, \cdots, n\}$ and in $G^{(n-k)}$ at 0, with one-sided derivatives as in (3.1.4).

Next, we observe from (3.1.5) that the spline $G$ in (3.1.7) is compactly supported, with support interval given by (3.1.3), if and only if the coefficient sequence $\{\alpha(j) : \ j = 0, \cdots, n\}$ satisfies the identity

$$\frac{x^{n-k}}{(n-k)!} + \sum_{j=0}^{n} \alpha(j)(x-j)^n = 0, \quad x \in \mathbb{R}, \tag{3.1.8}$$

or equivalently,

$$\left(\frac{d}{dx}\right)^l \left[\frac{x^{n-k}}{(n-k)!} + \sum_{j=0}^{n} \alpha(j)(x-j)^n\right]\Bigg|_{x=0} = 0, \quad l = 0, \cdots, n, \tag{3.1.9}$$

that is,

$$\sum_{j=0}^{n} j^{n-l}\alpha(j) = \frac{(-1)^{k+1}k!}{n!}\delta(n-k-l), \quad l = 0, \cdots, n, \tag{3.1.10}$$

which for convenience we now rewrite in the form

$$\sum_{j=0}^{n} j^l\alpha(j) = \frac{(-1)^{k+1}k!}{n!}\delta(l-k), \quad l = 0, \cdots, n. \tag{3.1.11}$$

Observe that the coefficient matrix $M$ of the linear system (3.1.11) is the $(n+1)\times(n+1)$ matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \\ 0 & 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1^n & 2^n & \cdots & n^n \end{bmatrix}, \tag{3.1.12}$$

of which the transpose $M^T$ is the Vandermonde matrix with respect to the (distinct) point set $\{0, 1, \cdots, n\}$, and therefore (see e.g. [38]) an invertible matrix. Hence $M = (M^T)^T$ is also an invertible matrix, according to which there exists precisely one solution $\{\alpha(j) = \alpha_{n,k}(j) : \ j = 0, \cdots, n\}$ of the linear system (3.1.11), which we claim to be explicitly given by the formulation

$$\alpha(j) = \alpha_{n,k}(j) = \frac{(-1)^{k+1}}{n!} L_{n,j}^{(k)}(0), \quad j = 0, \cdots, n, \tag{3.1.13}$$

with $\{L_{n,j} : \ j = 0, \cdots, n\} \subset \pi_n$ denoting the Lagrange fundamental polynomials of degree $n$ with respect to the interpolation point set $\{0, 1, \cdots, n\}$, as given by

$$L_{n,j}(x) := \prod_{j\neq\mu=0}^{n} \left(\frac{x-\mu}{j-\mu}\right), \quad j = 0, \cdots, n, \tag{3.1.14}$$

according to which

$$L_{n,j}(l) = \delta(l-j), \quad j, l = 0, \cdots, n. \tag{3.1.15}$$

Also, we have the identity (see e.g. [38])

$$\sum_{j=0}^{n} j^l \ L_{n,j}(x) = x^l, \quad x \in \mathbb{R}, \quad l = 0, \cdots, n, \tag{3.1.16}$$

from which we deduce that

$$\sum_{j=0}^{n} j^l \ L_{n,j}^{(k)}(0) = k!\delta(l-k), \quad l = 0, \cdots, n; \quad k = 0, 1, \cdots . \tag{3.1.17}$$

It follows from (3.1.17) that the (unique) solution of the linear system (3.1.11) is indeed given by the formula (3.1.13). Since, moreover,

$$\prod_{j\neq\mu=0}^{n} (j-\mu) = \frac{(-1)^{n-j} n!}{\binom{n}{j}}, \quad j = 0, \cdots, n, \tag{3.1.18}$$

we have therefore established the following result.

**Theorem 3.1.1** *For any integer $n \in \mathbb{N}$, let the spline sequence $\{G_{n,k} : \ k = 1, \cdots, \sigma_n\}$, with the integer $\sigma_n$ given by (3.1.1), be defined by*

$$G_{n,k}(x) := \frac{x_+^{n-k}}{(n-k)!} + \frac{(-1)^{n-1-k}\ k!}{(n!)^2} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \lambda_{n,k}(j)(x-j)_+^n, \quad x \in \mathbb{R}, \quad k = 1, \cdots, \sigma_n, \tag{3.1.19}$$

*where*

$$\sum_{l=0}^{n} \lambda_{n,l}(j) x^l := \prod_{j \neq \mu = 0}^{n} (x - \mu), \quad j = 0, \cdots, n. \tag{3.1.20}$$

*Then, for any $k \in \{1, \cdots, \sigma_n\}$, $G_{n,k}$ is a compactly supported spline in $S_{n,k}(\mathbb{Z})$, with*

$$\text{supp } G_{n,k} = [0, n], \tag{3.1.21}$$

*and with the only discontinuities in $G_{n,k}$ and its derivatives occurring in $G_{n,k}^{(n)}$ at all of the points $\{0, \cdots, n\}$, and in $G_{n,k}^{(n-k)}$ at 0, with one-sided derivatives*

$$G_{n,k}^{(n-k)}(0_-) = 0; \qquad G_{n,k}^{(n-k)}(0_+) = 1. \tag{3.1.22}$$

**Example 3.1.1** For $n = 1$ and $k = 1$, it follows from (3.1.20) that

$$\sum_{k=0}^{1} \lambda_{1,k}(j) x^k = \begin{cases} x - 1, & j = 0; \\ x, & j = 1, \end{cases} \tag{3.1.23}$$

and thus

$$\lambda_{1,1}(j) = \begin{cases} 1, & j = 0; \\ 1, & j = 1, \end{cases} \tag{3.1.24}$$

which can now be inserted into (3.1.19) to obtain

$$G_{1,1}(x) = x_+^0 - x_+ + (x-1)_+ = \begin{cases} 1 - x, & x \in [0, 1); \\ 0, & x \in \mathbb{R} \setminus [0, 1). \end{cases} \tag{3.1.25}$$

Note from (3.1.25) and (1.3.16) in Example 1.3.2 that $G_{1,1} = \phi_2$, the graph of which is given in Fig. 1.3(b). ■

**Example 3.1.2** For $n = 2$ and $k = 1$, it follows from (3.1.20) that

$$\sum_{k=0}^{2} \lambda_{2,k}(j) x^k = \begin{cases} (x-1)(x-2) = x^2 - 3x + 2, & j = 0; \\ x(x-2) = x^2 - 2x, & j = 1; \\ x(x-1) = x^2 - x, & j = 2, \end{cases} \tag{3.1.26}$$

and thus

$$\lambda_{2,1}(j) = \begin{cases} -3, & j = 0; \\ -2, & j = 1; \\ -1, & j = 2, \end{cases} \tag{3.1.27}$$

which we can now insert into (3.1.19) to obtain

$$G_{2,1}(x) = x_+ - \frac{3}{4}x_+^2 + (x-1)_+^2 - \frac{1}{4}(x-2)_+^2, \quad x \in \mathbb{R}, \tag{3.1.28}$$

that is,

$$G_{2,1}(x) = \begin{cases} -\frac{3}{4}x^2 + x, & x \in [0,1); \\ \frac{1}{4}x^2 - x + 1, & x \in [1,2); \\ 0, & x \in \mathbb{R} \setminus [0,2), \end{cases} \tag{3.1.29}$$

the graph of which is shown in Fig. 3.1. ■

By continuing similarly, we obtain Table 3.1, in which the splines $\{G_{n,k} : \ k = 1, \cdots, \sigma_n\}$ are given explicitly for $n = 1, \cdots, 5$, for which the corresponding graphs for $n = 2, \cdots, 5$ are shown in Fig. 3.1-3.4.

*Table 3.1: The splines* $\{G_{n,k} : \ k = 1, \cdots, \sigma_n\}$ *for* $n = 1, \cdots, 5$

| $n$ | $k$ | $G_{n,k}$ |
|---|---|---|
| 1 | 1 | $(1-x)\chi_{[0,1)}(x)$ |
| 2 | 1 | $(-\frac{3}{4}x^2 + x)\chi_{[0,1)}(x) + (\frac{1}{4}x^2 - x + 1)\chi_{[1,2)}(x)$ |
| 3 | 1 | $(-\frac{11}{36}x^3 + \frac{1}{2}x^2)\chi_{[0,1)}(x) + (\frac{7}{36}x^3 - x^2 + \frac{3}{2}x - \frac{1}{2})\chi_{[1,2)}(x)$ $+(-\frac{1}{18}x^3 + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{3}{2})\chi_{[2,3)}(x)$ |
| 3 | 2 | $(-\frac{1}{3}x^3 + x)\chi_{[0,1)}(x) + (\frac{1}{2}x^3 - \frac{5}{2}x^2 + \frac{7}{2}x - \frac{5}{6})\chi_{[1,2)}(x)$ $+(-\frac{1}{6}x^3 + \frac{3}{2}x^2 - \frac{9}{2}x + \frac{9}{2})\chi_{[2,3)}(x)$ |
| 4 | 1 | $(-\frac{25}{288}x^4 + \frac{1}{6}x^3)\chi_{[0,1)}(x) + (\frac{23}{288}x^4 - \frac{1}{2}x^3 + x^2 - \frac{2}{3}x + \frac{1}{6})\chi_{[1,2)}(x)$ $+(-\frac{13}{288}x^4 + \frac{1}{2}x^3 - 2x^2 + \frac{10}{3}x - \frac{11}{6})\chi_{[2,3)}(x)$ $+(\frac{1}{96}x^4 - \frac{1}{6}x^3 + x^2 - \frac{8}{3}x + \frac{8}{3})\chi_{[3,4)}(x)$ |
| 4 | 2 | $(-\frac{35}{288}x^4 + \frac{1}{2}x^2)\chi_{[0,1)}(x) + (\frac{23}{96}x^4 - \frac{13}{9}x^3 + \frac{8}{3}x^2 - \frac{13}{9}x + \frac{13}{36})\chi_{[1,2)}(x)$ $+(-\frac{5}{32}x^4 + \frac{31}{18}x^3 - \frac{41}{6}x^2 + \frac{101}{9}x - \frac{215}{36})\chi_{[2,3)}(x)$ $+(\frac{11}{288}x^4 - \frac{11}{18}x^3 + \frac{11}{3}x^2 - \frac{88}{9}x + \frac{88}{9})\chi_{[3,4)}(x)$ |
| 4 | 3 | $(-\frac{5}{48}x^4 + x)\chi_{[0,1)}(x) + (\frac{13}{48}x^4 - \frac{3}{2}x^3 + \frac{9}{4}x^2 - \frac{1}{2}x + \frac{3}{8})\chi_{[1,2)}(x)$ $+(-\frac{11}{48}x^4 + \frac{5}{2}x^3 - \frac{39}{4}x^2 + \frac{31}{2}x - \frac{61}{8})\chi_{[2,3)}(x)$ $+(\frac{1}{16}x^4 - x^3 + 6x^2 - 16x + 16)\chi_{[3,4)}(x)$ |

| $n$ | $k$ | $G_{n,k}$ |
|---|---|---|
| 5 | 1 | $\left(-\frac{137}{7200}x^5+\frac{1}{24}x^4\right)\chi_{[0,1)}(x)+\left(\frac{163}{7200}x^5-\frac{1}{6}x^4+\frac{5}{12}x^3-\frac{5}{12}x^2+\frac{5}{24}x-\frac{1}{24}\right)\chi_{[1,2)}(x)$ $+\left(\frac{137}{7200}x^5+\frac{1}{4}x^4-\frac{5}{4}x^3+\frac{35}{12}x^2-\frac{25}{8}x+\frac{31}{24}\right)\chi_{[2,3)}(x)$ $+\left(\frac{7}{800}x^5-\frac{1}{6}x^4+\frac{5}{4}x^3-\frac{55}{12}x^2+\frac{65}{8}x-\frac{131}{24}\right)\chi_{[3,4)}(x)$ $+\left(-\frac{1}{600}x^5+\frac{1}{24}x^4-\frac{5}{12}x^3+\frac{25}{12}x^2-\frac{125}{24}x+\frac{125}{24}\right)\chi_{[4,5)}(x)$ |
| 5 | 2 | $\left(-\frac{1}{32}x^5+\frac{1}{6}x^3\right)\chi_{[0,1)}(x)+\left(\frac{109}{1440}x^5-\frac{77}{144}x^4+\frac{89}{72}x^3-\frac{77}{72}x^2+\frac{77}{144}x-\frac{77}{720}\right)\chi_{[1,2)}(x)$ $+\left(-\frac{7}{96}x^5+\frac{137}{144}x^4-\frac{113}{24}x^3+\frac{779}{72}x^2-\frac{545}{48}x+\frac{3347}{720}\right)\chi_{[2,3)}(x)$ $+\left(\frac{17}{480}x^5-\frac{97}{144}x^4+\frac{121}{24}x^3-\frac{1327}{72}x^2+\frac{1561}{48}x-\frac{15607}{720}\right)\chi_{[3,4)}(x)$ $+\left(-\frac{1}{144}x^5+\frac{25}{144}x^4-\frac{125}{72}x^3+\frac{625}{72}x^2-\frac{3125}{144}x+\frac{3125}{144}\right)\chi_{[4,5)}(x)$ |
| 5 | 3 | $\left(-\frac{17}{480}x^5+\frac{1}{2}x^2\right)\chi_{[0,1)}(x)+\left(\frac{9}{80}x^5-\frac{71}{96}x^4+\frac{71}{48}x^3-\frac{47}{48}x^2+\frac{71}{96}x-\frac{71}{480}\right)\chi_{[1,2)}(x)$ $+\left(-\frac{2}{15}x^5+\frac{55}{32}x^4-\frac{401}{48}x^3+\frac{299}{16}x^2-\frac{1817}{96}x+\frac{247}{32}\right)\chi_{[2,3)}(x)$ $+\left(\frac{17}{240}x^5-\frac{43}{32}x^4+\frac{481}{48}x^3-\frac{583}{16}x^2+\frac{6121}{96}x-\frac{6703}{160}\right)\chi_{[3,4)}(x)$ $+\left(-\frac{7}{480}x^5+\frac{35}{96}x^4-\frac{175}{48}x^3+\frac{875}{48}x^2-\frac{4375}{96}x+\frac{4375}{96}\right)\chi_{[4,5)}(x)$ |
| 5 | 4 | $\left(-\frac{1}{40}x^5+x\right)\chi_{[0,1)}(x)+\left(\frac{11}{120}x^5-\frac{7}{12}x^4+\frac{7}{6}x^3-\frac{7}{6}x^2+\frac{19}{12}x-\frac{7}{60}\right)\chi_{[1,2)}(x)$ $+\left(-\frac{1}{8}x^5+\frac{19}{12}x^4-\frac{15}{2}x^3+\frac{97}{6}x^2-\frac{63}{4}x+\frac{409}{60}\right)\chi_{[2,3)}(x)$ $+\left(\frac{3}{40}x^5-\frac{17}{12}x^4+\frac{21}{2}x^3-\frac{227}{6}x^2+\frac{261}{4}x-\frac{2507}{60}\right)\chi_{[3,4)}(x)$ $+\left(-\frac{1}{60}x^5+\frac{5}{12}x^4-\frac{25}{6}x^3+\frac{125}{6}x^2-\frac{625}{12}x+\frac{625}{12}\right)\chi_{[4,5)}(x)$ |

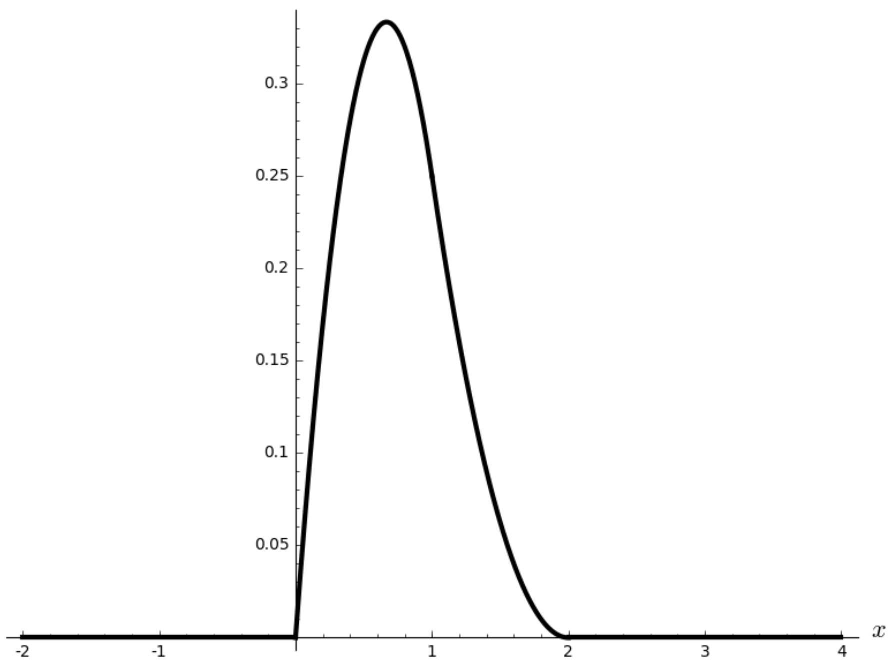



**Figure 3.1:** *Graph of* $G_{2,1}$

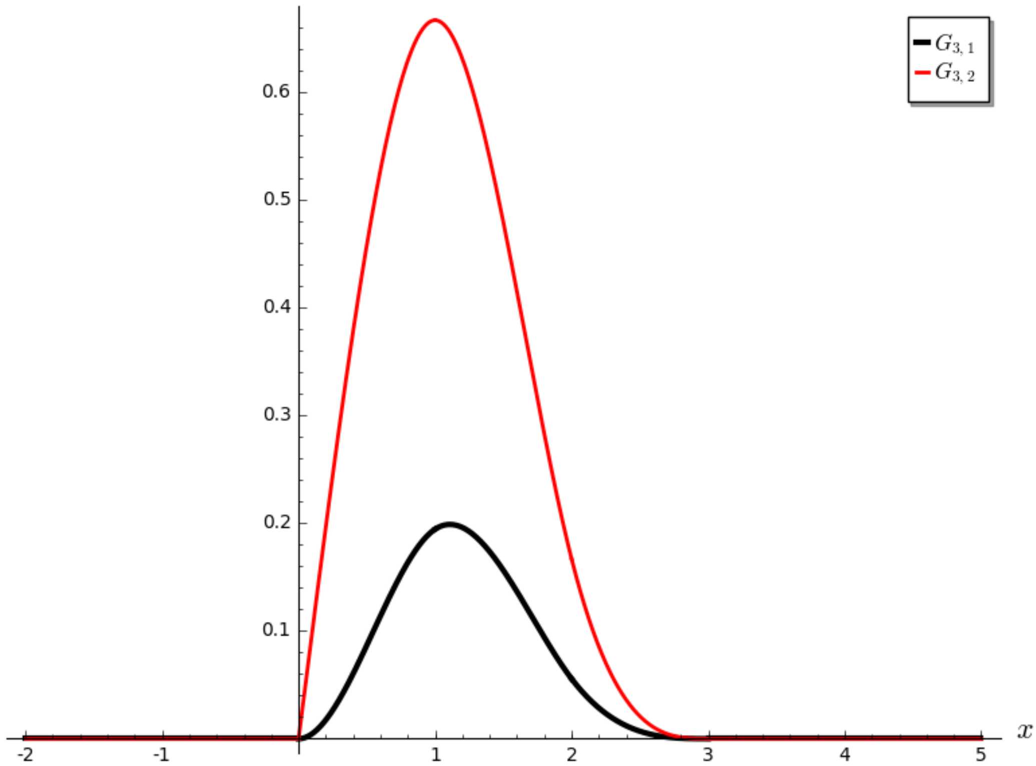



**Figure 3.2:** *Graphs of* $G_{3,1}$ *and* $G_{3,2}$

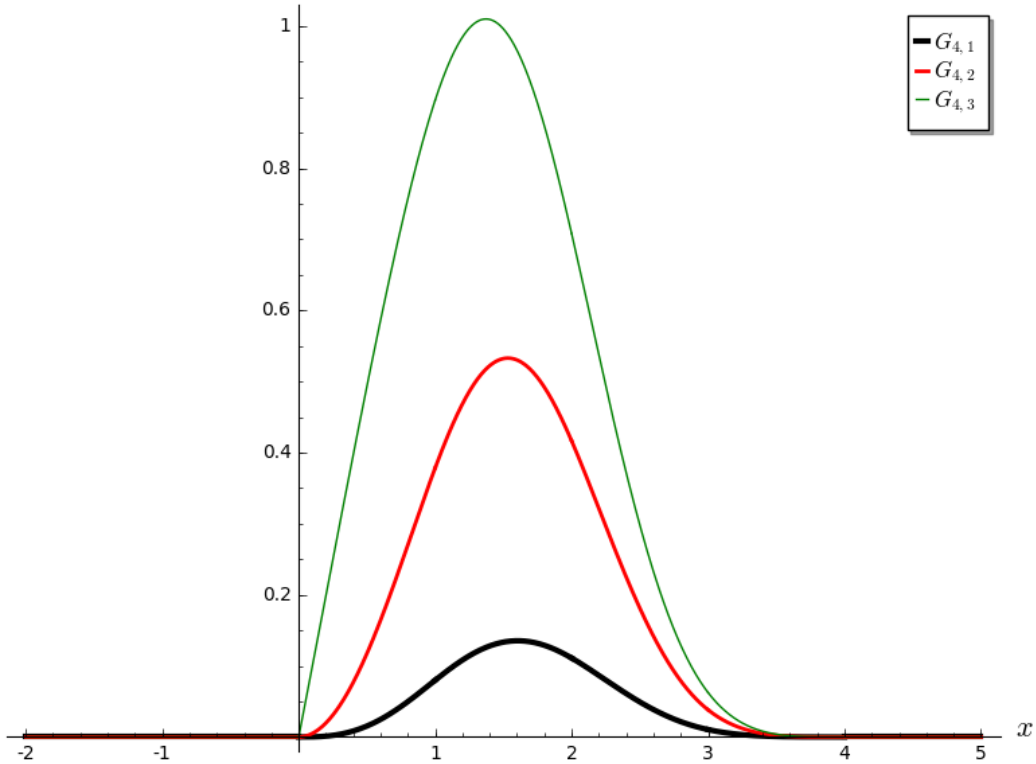



**Figure 3.3:** *Graphs of* $G_{4,1}$, $G_{4,2}$ *and* $G_{4,3}$

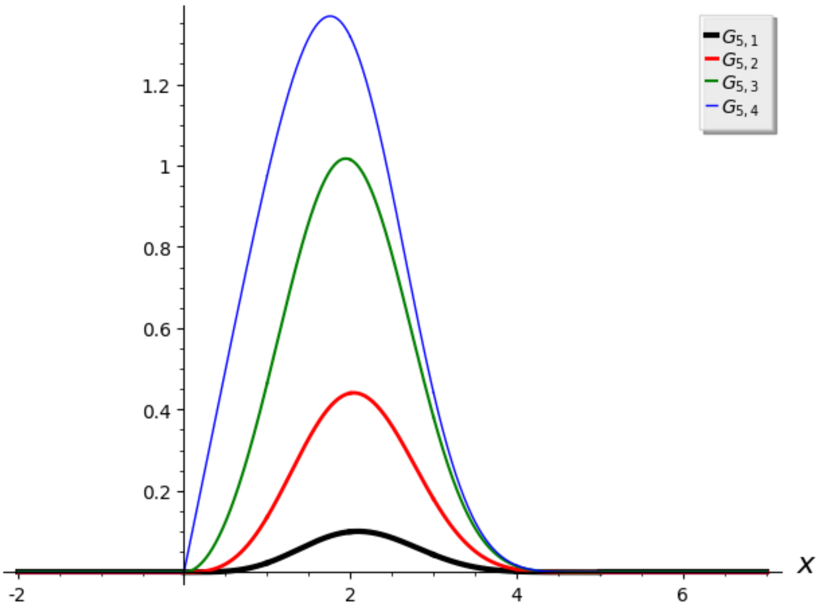



**Figure 3.4:** *Graphs of* $G_{5,1}$, $G_{5,2}$, $G_{5,3}$ *and* $G_{5,4}$

Based on Theorem 3.1.1, we can now state the following vector spline result.

**Theorem 3.1.2** *For any integers* $\nu \geq 2$ *and* $n \geq \rho_\nu$, *where*

$$\rho_\nu := \begin{cases} 1, & \nu = 2; \\ \nu, & \nu \geq 3, \end{cases} \tag{3.1.30}$$

*let the vector spline* $\mathbf{\Phi}_{\nu,n} : \mathbb{R} \to \mathbb{R}^\nu$ *be defined by*

$$\mathbf{\Phi}_{\nu,n} := (N_n, G_{n,1}, \cdots, G_{n,\nu-1})^T, \tag{3.1.31}$$

*with* $N_n$ *denoting the* $n^{th}$ *degree cardinal B-spline in* (1.3.3), (1.3.4), *and where the splines* $\{G_{n,k} : k = 1, \cdots, \nu - 1\}$ *are given as in Theorem* 3.1.1. *Then*

$$\mathbf{\Phi}_{\nu,n} \in \mathbf{C}_0^{n-\nu}(\mathbb{R}), \tag{3.1.32}$$

*with*

$$\left.\begin{aligned} &\operatorname{supp} N_n = [0, n+1]; \\ &\operatorname{supp} G_{n,k} = [0, n], \quad k = 1, \cdots, \nu - 1. \end{aligned}\right\} \tag{3.1.33}$$

**Remark 3.1.1** (a) Observe from (3.1.32) that, for any given vector length $\nu$ of $\mathbf{\Phi}_{\nu,n}$, the smoothness of $\mathbf{\Phi}_{\nu,n}$ can be increased arbitrarily by choosing the spline degree $n$ sufficiently large. In [31], only the choice $n = \nu$ was considered, in which case, from

Theorem 3.3.1, the spline component $G_{n,\nu-1} = G_{\nu,\nu-1}$ has a derivative with a jump discontinuity at 0.

(b) By recalling also (3.1.25) in Example 3.1.1, note that the vector spline $\mathbf{\Phi}_{2,1}$ corresponds precisely with the vector function $\mathbf{\Phi} = (\phi_1, \phi_2)^T$ defined by (1.3.16) of Example 3.1.2, and with graph shown in Fig. 1.3.

Selected graphs of the vector splines $\mathbf{\Phi}_{\nu,n}$ are shown in Fig. 3.5-3.10.

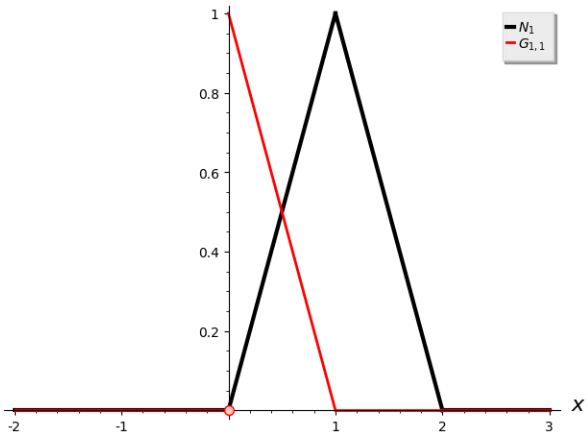



**Figure 3.5:** *Graph of* $\mathbf{\Phi}_{2,1} = (N_1, G_{1,1})^T$

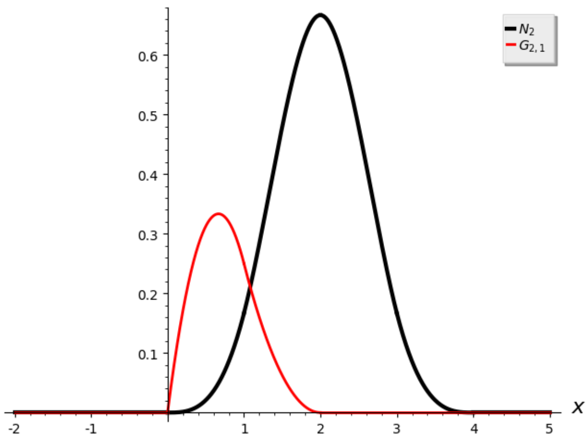



**Figure 3.6:** *Graph of* $\mathbf{\Phi}_{2,2} = (N_2, G_{2,1})^T$

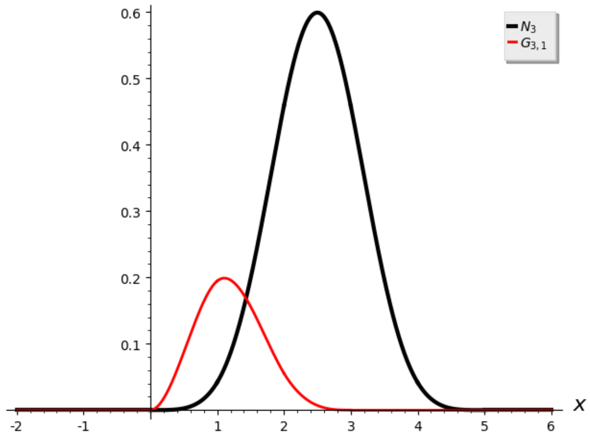



**Figure 3.7:** *Graph of* $\mathbf{\Phi}_{2,3} = (N_3, G_{3,1})^T$

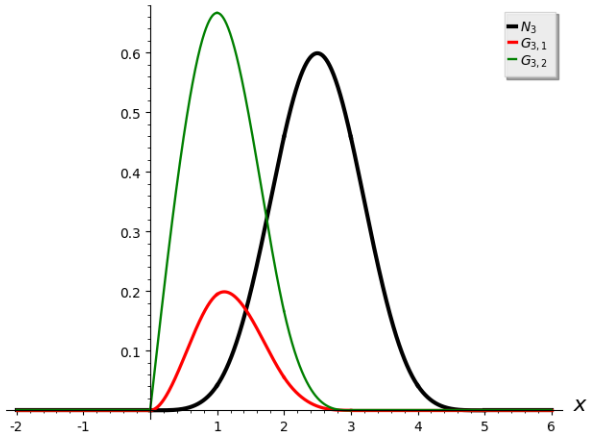



**Figure 3.8:** *Graph of* $\mathbf{\Phi}_{3,3} = (N_3, G_{3,1}, G_{3,2})^T$

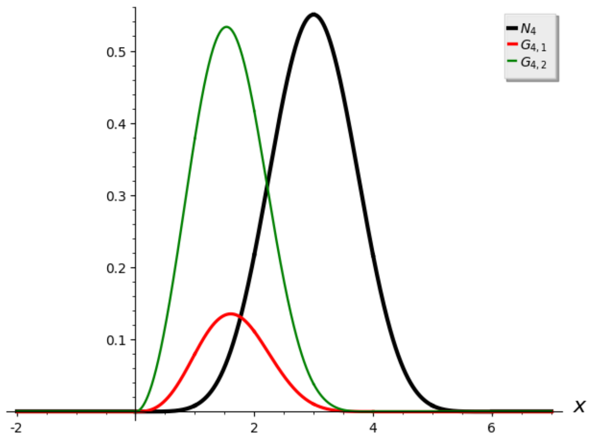



**Figure 3.9:** *Graph of* $\mathbf{\Phi}_{3,4} = (N_4, G_{4,1}, G_{4,2})^T$

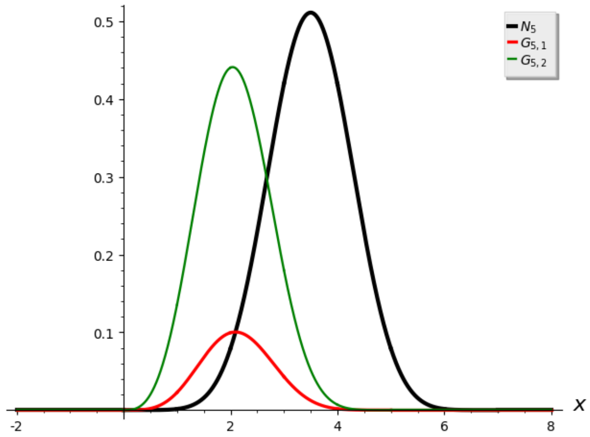



**Figure 3.10:** *Graph of* $\mathbf{\Phi}_{3,5} = (N_5, G_{5,1}, G_{5,2})^T$

We proceed in the next two sections to establish, by means of a Fourier transform technique, the refinability of the vector spline

$$\mathbf{\Phi}_{\nu,n} := (N_n, G_{n,1}, \cdots, G_{n,\nu-1})^T, \tag{3.1.34}$$

with $N_n$ denoting the cardinal spline of degree $n$, as defined recursively in (1.3.3)-(1.3.4).

Observe in particular, by recalling also (3.1.25) in Example 3.1.1, that the refinability of the vector spline $\mathbf{\Phi}_{2,1} = (N_1, G_{1,1})^T$ has already been shown in Example 1.3.2, with explicitly calculated matrix refinement sequence $\{P(k)\}$ as in (1.3.17).

## 3.2 Fourier transform formulations

The first component spline of the vector spline $\mathbf{\Phi}_{\nu,n}$ in (3.1.34), namely the cardinal B-spline $N_n$, has the following explicitly formulated Fourier transform, as was established in [2].

**Theorem 3.2.1** *For any non-negative integer $n$, the cardinal B-spline $N_n$ of degree $n$, as recursively defined in* (1.3.3), (1.3.4), *has the Fourier transform*

$$\widehat{N}_n(\omega) = \begin{cases} \left(\dfrac{1-e^{-i\omega}}{i\omega}\right)^{n+1}, & \omega \in \mathbb{C} \setminus \{0\}; \\ 1, & \omega = 0. \end{cases} \tag{3.2.1}$$

**Proof.** First, for $n = 0$, we deduce from (2.3.5) and (1.3.3) that

$$\widehat{N}_0(\omega) = \int_0^1 e^{-i\omega x}\,dx = \begin{cases} \dfrac{1-e^{-i\omega}}{i\omega}, & \omega \in \mathbb{C}\setminus\{0\}; \\ 1, & \omega = 0, \end{cases} \tag{3.2.2}$$

which shows that (3.2.1) holds for $n = 0$. Next, we apply (2.3.5) and (1.3.4) to deduce that, for any fixed non-negative integer $n$, and any $\omega \in \mathbb{C}$,

$$\begin{aligned} \widehat{N}_{n+1}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\Big[\int_0^1 N_n(x-t)\,dx\Big]\,dt &= \int_0^1 \Big[\int_{-\infty}^{\infty} e^{-i\omega x} N_n(x-t)\,dt\Big]\,dx \\ &= \int_0^1 \Big[\int_{-\infty}^{\infty} e^{-i\omega(x+t)} N_n(x)\,dx\Big]\,dt \\ &= \Big[\int_{-\infty}^{\infty} e^{-i\omega x} N_n(x)\,dx\Big]\Big[\int_0^1 e^{-i\omega t}\,dt\Big], \end{aligned}$$

thus

$$\widehat{N}_{n+1}(\omega) = \widehat{N}_n(\omega)\begin{cases} \dfrac{1-e^{-i\omega}}{i\omega}, & \omega \in \mathbb{C}\setminus\{0\}; \\ 1, & \omega = 0, \end{cases}$$

which, together with (3.2.2), proves (3.2.1) inductively. ■

Our next step is to obtain an explicit formulation of the Fourier transform $\widehat{G}_{n,k}$ of the spline $G_{n,k}$ of Theorem 3.1.1. We shall rely on the following power series result with respect to the principal branch natural logarithmic function Log $z$ in the complex plane.

**Theorem 3.2.2** *For any $k \in \mathbb{N}$,*

$$(\text{Log } z)^k = \sum_{j=k}^{\infty} t_k(j)(z-1)^j, \quad |z-1| < 1, \tag{3.2.3}$$

*where the real coefficient sequence $\{t_k(j) : j = k, k+1, \cdots\}$ is given by*

$$t_k(j) := (-1)^{k-j}\begin{cases} \dfrac{1}{j}, \quad k = 1; \\ \displaystyle\sum_{m=1}^{j-1} \frac{1}{m(j-m)}, \quad k = 2; \\ \displaystyle\sum_{m_1=1}^{j-(k-1)} \frac{1}{m_1} \sum_{m_2=1}^{j-(k-2)-m_1} \frac{1}{m_2} \cdots \\ \qquad \displaystyle\sum_{m_{k-2}=1}^{j-2-m_1-\cdots-m_{k-3}} \frac{1}{m_{k-2}} \sum_{m_{k-1}=1}^{j-1-m_1-\cdots-m_{k-2}} \frac{1}{m_{k-1}(j-m_1-\cdots-m_{k-1})}, \\ \qquad\qquad k \geq 3. \end{cases} \tag{3.2.4}$$

**Proof.** After first noting that the case $k = 1$ is a standard power series result (see e.g. [39]), we next deduce that, for any $|z-1| < 1$,

$$\begin{aligned}
(\text{Log } z)^2 &= \Big\{\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}(z-1)^m\Big\}\Big\{\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(z-1)^j\Big\} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m}\Big\{\sum_{j=1}^{\infty} \frac{(-1)^j}{j}(z-1)^{j+m}\Big\} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{j=m+1}^{\infty} \frac{(-1)^{j-m}}{j-m}(z-1)^j \\
&= \sum_{j=2}^{\infty}(-1)^j\Big\{\sum_{m=1}^{j-1} \frac{1}{m(j-m)}\Big\}(z-1)^j,
\end{aligned}$$

which proves (3.2.3),(3.2.4) for $k = 2$.

For $k \geq 3$, we have, for any $|z-1| < 1$,

$$\begin{aligned}
(\text{Log } z)^k &= \Big\{\sum_{m_1=1}^{\infty} \frac{(-1)^{m_1+1}}{m_1}(z-1)^{m_1}\Big\}\cdots\Big\{\sum_{m_{k-1}=1}^{\infty} \frac{(-1)^{m_{k-1}+1}}{m_{k-1}}(z-1)^{m_{k-1}}\Big\}\Big\{\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(z-1)^j\Big\} \\
&= \sum_{m_1=1}^{\infty} \frac{(-1)^{m_1+1}}{m_1} \cdots \sum_{m_{k-1}=1}^{\infty} \frac{(-1)^{m_{k-1}+1}}{m_{k-1}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(z-1)^{m_1+\cdots+m_{k-1}+j}.
\end{aligned}$$

Then,

$$\begin{aligned}
(\text{Log } z)^k &= \sum_{m_1=1}^{\infty} \frac{(-1)^{m_1+1}}{m_1} \cdots \sum_{m_{k-1}=1}^{\infty} \frac{(-1)^{m_{k-1}+1}}{m_{k-1}} \sum_{j=m_1+\cdots+m_{k-1}+1}^{\infty} \frac{(-1)^{j-m_1-\cdots-m_{k-1}+1}}{j-m_1-\cdots-m_{k-1}}(z-1)^j \\
&= (-1)^k \sum_{m_1=1}^{\infty} \frac{1}{m_1} \cdots \sum_{j=m_1+\cdots+m_{k-2}+2}^{\infty} (-1)^j \sum_{m_{k-1}=1}^{j-1-m_1-\cdots-m_{k-2}} \frac{1}{m_{k-1}(j-m_1-\cdots-m_{k-1})}(z-1)^j \\
&= \quad \cdots \\
&= (-1)^k \sum_{j=k}^{\infty}(-1)^j\Bigg\{\sum_{m_1=1}^{j-(k-1)} \frac{1}{m_1} \sum_{m_2=1}^{j-(k-2)-m_1} \frac{1}{m_2} \cdots \\
&\qquad \sum_{m_{k-2}=1}^{j-2-m_1-\cdots-m_{k-3}} \frac{1}{m_{k-2}} \sum_{m_{k-1}=1}^{j-1-m_1-\cdots-m_{k-2}} \frac{(z-1)^j}{m_{k-1}(j-m_1-\cdots-m_{k-1})}\Bigg\}(z-1)^j,
\end{aligned}$$

and thereby proving (3.2.3), (3.2.4) for $k \geq 3$. ■

We proceed to explicitly formulate the Fourier transforms of the splines in Theorem 3.1.1.

**Theorem 3.2.3** *For any integers $n \in \mathbb{N}$ and $k \in \{1, \cdots, \sigma_n\}$, with $\sigma_n$ defined by* (3.1.1)*, the Fourier transform of the spline $G_{n,k}$ in Theorem* 3.1.1 *has the formulation*

$$\widehat{G}_{n,k}(\omega) = \begin{cases} (-1)^k \dfrac{(-i\omega)^k - T_{n,k}(e^{-i\omega})}{(i\omega)^{n+1}}, & \omega \in \mathbb{C} \setminus \{0\}; \\ \int_0^n G_{n,k}(x)\, dx, \quad \omega = 0, \end{cases} \tag{3.2.5}$$

*where $T_{n,k}$ is the polynomial in $\pi_n$ given by*

$$T_{n,k}(z) := \sum_{j=k}^{n} t_k(j)(z-1)^j, \tag{3.2.6}$$

*with the real coefficient sequence $\{t_k(j) : \ j = k, \cdots, n\}$ defined as in* (3.2.4).

**Remark 3.2.1** Observe from (3.2.3) and (3.2.6) that $T_{n,k}$ is the Taylor polynomial of degree $n$ with respect to $z = 1$ of the function $(\text{Log}\ \ z)^k$. ∎

## Proof of Theorem 3.2.3.

First, for $n = 1$ and $k = 1$, we may apply (3.1.25) and (2.3.5) to deduce by means of integration by parts that (3.2.5) is satisfied, with, from (3.2.6) and (3.2.4), $T_{1,1}(z) = z-1$. Suppose next that $n \geq 2$ and $k \in \{1, \cdots, n-1\}$, from which it then follows from Theorem 3.1.1 that

$$G_{n,k}\Big|_{[0,n]} \in C^{n-1}[0,n], \tag{3.2.7}$$

with

$$G_{n,k}^{(l)}(0) = 0, \quad l \in \{0, \cdots, n-1\} \setminus \{n-k\}; \tag{3.2.8}$$

$$G_{n,k}^{(l)}(n) = 0, \quad l = 0, \cdots, n-1. \tag{3.2.9}$$

Let $\omega \in \mathbb{C} \setminus \{0\}$. It follows from (2.3.5), (3.1.3), (3.2.7), (3.2.8) and (3.2.9), together with integration by parts, that

$$\begin{aligned}\widehat{G}_{n,k}(\omega) = \int_0^n G_{n,k}(x) e^{-i\omega x}\ dx &= \frac{1}{i\omega}\int_0^n G'_{n,k}(x) e^{-i\omega x}\ dx \\ &= \quad \cdots \\ &= \frac{1}{(i\omega)^{n-k}} \int_0^n G_{n,k}^{(n-k)}(x) e^{-i\omega x}\ dx \\ &= \frac{1}{(i\omega)^{n+1-k}} \left[1 + \int_0^n G_{n,k}^{(n+1-k)}(x) e^{-i\omega x}\ dx\right], \end{aligned} \tag{3.2.10}$$

by virtue also of the second equation in (3.1.22). A similar procedure based on (3.2.7), (3.2.8) and (3.2.9) now yields

$$\begin{aligned}\int_0^n G_{n,k}^{(n+1-k)}(x) e^{-i\omega x}\ dx &= \frac{1}{i\omega}\int_0^n G_{n,k}^{(n+2-k)}(x) e^{-i\omega x}\ dx \\ &= \cdots = \frac{1}{(i\omega)^{k-2}} \int_0^n G_{n,k}^{(n-1)}(x) e^{-i\omega x}\ dx. \end{aligned} \tag{3.2.11}$$

According to (3.1.19) and (3.1.5), $G_{n,k}^{(n-1)}$ is a continuous piecewise linear function with respect to the integer partition $\mathbb{Z}$ of $\mathbb{R}$. Hence both $G_{n,k}^{(n-1)}$ and $e^{-i\omega x}$ are Lipschitz continuous, and thus absolutely continuous, and therefore also almost everywhere differentiable on $[0, n]$. Hence (see e.g. [40]) integration by parts may be performed one more time to yield, together with (3.2.8) and (3.2.9),

$$\int_0^n G_{n,k}^{(n-1)}(x)e^{-i\omega x}\, dx = \sum_{l=0}^{n-1} \int_l^{l+1} G_{n,k}^{(n)}(x)e^{-i\omega x} dx, \tag{3.2.12}$$

after having noted also that, according to (3.1.19) and (3.1.5), $G_{n,k}^{(n)}$ is a piecewise constant function with respect to the integer partition $\mathbb{Z}$ of $\mathbb{R}$, that is,

$$G_{n,k}^{(n)}(x) = c_{n,k,l}, \quad x \in [l, l+1),\ l = 0, \cdots, n-1, \tag{3.2.13}$$

for some real constants $\{c_{n,k,l} :\ l = 0, \cdots, n-1\}$. It follows from (3.2.12) and (3.2.13) that

$$\sum_{l=0}^{n-1} \int_l^{l+1} G_{n,k}^{(n)}(x)e^{-i\omega x}\, dx = \frac{1}{i\omega} \sum_{l=0}^{n-1} c_{n,k,l}\big[(e^{-i\omega})^l - (e^{-i\omega})^{l+1}\big]. \tag{3.2.14}$$

By combining (3.2.10), (3.2.11), (3.2.12) and (3.2.13), we deduce that the top line of (3.2.5) is satisfied, with the polynomial $T_{n,k} \in \pi_n$ given by

$$T_{n,k}(z) := (-1)^{k-1} \sum_{l=0}^{n-1} c_{n,k,l}(z^l - z^{l+1}). \tag{3.2.15}$$

Since the second line of (3.2.5) is an immediate consequence of (2.3.6), it therefore remains to show that the definitions (3.2.15) and (3.2.6) of the polynomial $T_{n,k}$ are equivalent.

To this end, we first observe from (3.2.5) that the polynomial $T_{n,k}$ in (3.2.15) satisfies

$$(-i\omega)^k - T_{n,k}(e^{-i\omega}) = (-1)^k (i\omega)^{n+1} \widehat{G}_{n,k}(\omega), \quad \omega \in \mathbb{C} \setminus \{0\}. \tag{3.2.16}$$

Since $n \geq 2$, it follows from Theorem 3.1.1 that $G_{n,k} \in C_0(\mathbb{R})$, so that we may apply the Paley-Wiener theorem (see e.g. [41]) to deduce that $\widehat{G}_{n,k}$ is a holomorphic function, so that we may differentiate (3.2.16) to obtain

$$\left(\frac{d}{d\omega}\right)^l \Big\{(-i\omega)^k - T_{n,k}(e^{-i\omega})\Big\}\Big|_{\omega=0} = 0, \quad l = 0, \cdots, n, \tag{3.2.17}$$

or equivalently,

$$\left(\frac{d}{d\omega}\right)^l \Big\{(\text{Log } z)^k - T_{n,k}(z)\Big\}\Big|_{z=1} = 0, \quad l = 0, \cdots, n, \text{ where } z = e^{-i\omega}. \tag{3.2.18}$$

We claim that (3.2.18) implies

$$\left(\frac{d}{dz}\right)^l \left\{(\text{Log } z)^k - T_{n,k}(z)\right\}\Big|_{z=1} = 0, \quad l = 0, \cdots, n, \text{ where } z = e^{-i\omega}. \tag{3.2.19}$$

Since (3.2.16) trivially implies (3.2.18) for $l = 0$, we proceed to prove (3.2.18) for $l = 1, \cdots, n$. To this end, we first show inductively that, for $l = 1, \cdots, n$, it holds that

$$\left(\frac{d}{dz}\right)^l = e^{il\omega} \sum_{j=1}^{l} c_l(j) \left(\frac{d}{d\omega}\right)^j, \quad \text{where } z = e^{-i\omega}, \tag{3.2.20}$$

for some coefficient sequence $\{c_l(j) : \ j = 1, \cdots, l\} \subset \mathbb{C}$. To prove (3.2.20) for $l = 1$, we note that $z = e^{-i\omega}$ yields

$$\frac{d}{d\omega} = \frac{dz}{d\omega} \frac{d}{dz} = -ie^{-i\omega} \ \frac{d}{dz},$$

and thus

$$\frac{d}{dz} = ie^{i\omega} \ \frac{d}{d\omega}, \tag{3.2.21}$$

according to which (3.2.20) holds for $l = 1$, with $c_1(1) = i$.

Suppose next that, for any fixed $l \in \{1, \cdots, n\}$, equation (3.2.20) is satisfied for some coefficient sequence $\{c_l(j) : \ j = 1, \cdots, l\} \subset \mathbb{C}$. By applying also (3.2.21), we deduce that

$$\begin{aligned}
\left(\frac{d}{dz}\right)^{l+1} = \frac{d}{dz}\left(\frac{d}{dz}\right)^l &= ie^{i\omega}\frac{d}{d\omega}\left\{e^{il\omega}\sum_{j=1}^{l} c_l(j)\left(\frac{d}{d\omega}\right)^j\right\} \\
&= ie^{i\omega}\left\{ile^{il\omega}\sum_{j=1}^{l} c_l(j)\left(\frac{d}{d\omega}\right)^j + e^{il\omega}\sum_{j=1}^{l} c_l(j)\left(\frac{d}{d\omega}\right)^{j+1}\right\} \\
&= e^{i(l+1)\omega}\left\{-l\sum_{j=1}^{l} c_l(j)\left(\frac{d}{d\omega}\right)^j + i\sum_{j=1}^{l} c_l(j)\left(\frac{d}{d\omega}\right)^{j+1}\right\} \\
&= e^{i(l+1)\omega}\left\{-l\sum_{j=1}^{l} c_l(j)\left(\frac{d}{d\omega}\right)^j + i\sum_{j=2}^{l+1} c_l(j-1)\left(\frac{d}{d\omega}\right)^{j}\right\} \\
&= e^{i(l+1)\omega}\left\{\sum_{j=1}^{l+1} c_{l+1}(j)\left(\frac{d}{d\omega}\right)^j\right\},
\end{aligned}$$

where

$$c_{l+1}(j) := \begin{cases} -l \ c_l(1), \quad j = 1; \\ -l \ c_l(j) + i \ c_l(j-1), \quad j = 2, \cdots, l, \ (\text{if } l \geq 2); \\ i \ c_l(l), \quad j = l+1, \end{cases} \tag{3.2.22}$$

according to which (3.2.20) is also satisfied with $l$ replaced by $l+1$, and with the coefficient sequence $\{c_{l+1}(j): \; j = 1, \cdots, l+1\}$ given by (3.2.22). Our inductive proof of (3.2.20) is therefore complete.

The fact that (3.2.17) implies (3.2.19) also for $l = 1, \cdots, n$, is now an immediate consequence of (3.2.20).

Since $T_{n,k} \in \pi_n$, the condition (3.2.19) determines $T_{n,k}$ uniquely. By applying the power series result (3.2.3) in Theorem 3.2.2, we obtain

$$(\text{Log } z)^k - \sum_{j=k}^{n} t_k(j)(z-1)^j = \sum_{j=n+1}^{\infty} t_k(j)(z-1)^j, \quad |z-1| < 1, \tag{3.2.23}$$

according to which

$$\left(\frac{d}{dz}\right)^l \left[(\text{Log } z)^k - \sum_{j=k}^{n} t_k(j)(z-1)^j\right]\Big|_{z=1} = 0, \quad l = 0, \cdots, n, \tag{3.2.24}$$

so that we may deduce from (3.2.19) and (3.2.24) that $T_{n,k}$ is indeed the polynomial in $\pi_n$ defined by (3.2.6), (3.2.4), and thereby completing our proof. ■

Observe from (3.2.6) that, for any integers $n \in \mathbb{N}$ and $k \in \{1, \cdots, \sigma_n\}$, we have

$$T_{n,k}(z) = \sum_{j=k}^{n} t_k(j) \sum_{l=0}^{j} \binom{j}{l} z^l (-1)^{l-j} = \sum_{l=0}^{j} (-1)^l \left\{ \sum_{j=\max\{k,l\}}^{n} (-1)^j \binom{j}{l} t_k(j) \right\} z^l,$$

and thus

$$T_{n,k}(z) = \sum_{l=0}^{n} \tau_{n,k}(l) z^l, \tag{3.2.25}$$

where the coefficient sequence $\{\tau_{n,k}(l): \; l = 0, \cdots, n\}$ is given by

$$\tau_{n,k}(l) := (-1)^l \sum_{j=\max\{k,l\}}^{n} (-1)^j \binom{j}{l} t_k(j), \quad l = 0, \cdots, n, \tag{3.2.26}$$

with the sequence $\{t_k(j): \; j = k, \cdots, n\}$ defined as in (3.2.4).

Calculating by means of (3.2.25) and (3.2.4), we obtain the values in Table 3.2 for the coefficient sequences $\{\tau_{n,k}(l): \; l = 0, \cdots, n\}$ in (3.2.25), for $n \in \{1, \cdots, 5\}$.

*Table 3.2: Coefficients $\{\tau_{n,k}(l)\}$ of $T_{n,k}$, for $n \in \{1, \cdots, 5\}$*

| $n$ | $k$ | $\{\tau_{n,k}(l): \; l = 0, \cdots, n\}$ |
|---|---|---|
| 1 | 1 | $\{-1, 1\}$ |
| 2 | 1 | $\{-\frac{3}{2}, 2, -\frac{1}{2}\}$ |
| 3 | 1 | $\{-\frac{11}{6}, 3, -\frac{3}{2}, \frac{1}{3}\}$ |
| 3 | 2 | $\{2, -5, 4, -1\}$ |
| 4 | 1 | $\{-\frac{25}{12}, 4, -3, \frac{4}{3}, -\frac{1}{4}\}$ |
| 4 | 2 | $\{\frac{35}{12}, -\frac{26}{3}, \frac{19}{2}, -\frac{14}{3}, \frac{11}{12}\}$ |
| 4 | 3 | $\{-\frac{5}{2}, 9, -12, 7, -\frac{3}{2}\}$ |
| 5 | 1 | $\{-\frac{137}{60}, 5, -5, \frac{10}{3}, -\frac{5}{4}, \frac{1}{5}\}$ |
| 5 | 2 | $\{\frac{15}{4}, -\frac{77}{6}, \frac{107}{6}, -13, \frac{61}{12}, -\frac{5}{6}\}$ |
| 5 | 3 | $\{-\frac{17}{4}, \frac{71}{4}, -\frac{59}{2}, \frac{49}{2}, -\frac{41}{4}, \frac{7}{4}\}$ |
| 5 | 4 | $\{3, -14, 26, -24, 11, -2\}$ |

For any compactly supported function $f \in L^1(\mathbb{R})$ with Fourier transform $\widehat{f}$ satisfying the condition $\widehat{f}\Big|_{\mathbb{R}} \in L^1(\mathbb{R})$, the inverse Fourier transform $\mathcal{F}^{-1}\widehat{f}: \; \mathbb{R} \to \mathbb{C}$ is defined by

$$\big(\mathcal{F}^{-1}\widehat{f}\,\big)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)\; d\omega, \quad x \in \mathbb{R}. \tag{3.2.27}$$

We shall rely on the standard result (see e.g. [2]) which states that if the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R})$ also satisfies $\widehat{f}\Big|_{\mathbb{R}} \in L^1(\mathbb{R})$, then

$$f(x) = \big(\mathcal{F}^{-1}\widehat{f}\big)(x) \tag{3.2.28}$$

at each point $x$ where $f$ is continuous.

Since, for any $\omega \in \mathbb{R}$,

$$|1 - e^{-i\omega}| = |(1 - \cos\omega) + i\;\sin\omega| = 2|\sin(\omega/2)| \leq 2,$$

it follows from (3.2.1) that

$$|\widehat{N}_n(\omega)| \leq \frac{2}{|\omega|^{n+1}}, \quad \omega \in \mathbb{R} \setminus \{0\},$$

and thus

$$\widehat{N}_n \in L^1(\mathbb{R}), \quad n \geq 2. \tag{3.2.29}$$

Next, for any integers $n \geq 2$ and $k \in \{1, \cdots, n-1\}$, we may apply a standard Taylor remainder estimate in the complex plane (see e.g. [42]) to deduce from (3.2.23) and (3.2.6) that

$$\left|(-i\omega)^k - T_{n,k}(e^{-i\omega})\right| \leq K\,|1 - e^{-i\omega}|^{n+1}, \quad \omega \in \mathbb{R}, \tag{3.2.30}$$

which, together with (3.2.5) in Theorem 3.2.3, yields the estimate

$$|\widehat{G}_{n,k}(\omega)| \leq K\left|\frac{1 - e^{-i\omega}}{\omega}\right|^{n+1}, \quad \omega \in \mathbb{R} \setminus \{0\}, \tag{3.2.31}$$

from which we deduce that, similarly to the argument leading to (3.2.29), we have

$$\widehat{G}_{n,k} \in L^1(\mathbb{R}), \quad n \geq 2. \tag{3.2.32}$$

Since also $N_n,\ G_{n,k} \in C(\mathbb{R})$, we now apply (3.2.29) and (3.2.32) to immediately deduce the following result from (3.2.28).

**Theorem 3.2.4** *For any $n \in \mathbb{N}$, we have, in the notation of* (1.3.3), (1.3.4) *and Theorem* 3.1.1, *the inverse Fourier transform results*

$$N_n(x) = \left(\mathcal{F}^{-1}\widehat{N}_n\right)(x), \quad x \in \mathbb{R}; \tag{3.2.33}$$

$$G_{n,k}(x) = \left(\mathcal{F}^{-1}\widehat{G}_{n,k}\right)(x), \quad x \in \mathbb{R}, \quad k = 1, \cdots, \sigma_n. \tag{3.2.34}$$

## 3.3 Vector spline refinability

In this section, we apply the spline Fourier transforms of Section 3.2 to establish the refinability of the vector spline $\mathbf{\Phi}_{\nu,n}$ of Theorem 3.1.2.

We shall rely on the following general equivalent Fourier transform formulation of the vector refinement equation (1.3.1).

**Theorem 3.3.1** *For any integer $\nu \in \mathbb{N}$, let $\{\phi_1, \cdots, \phi_\nu\}$ denote a set of compactly supported functions in $L^1(\mathbb{R})$ such that the inverse Fourier transform results*

$$\phi_j(x) = \left(\mathcal{F}^{-1}\widehat{\phi}_j\right)(x), \quad x \in \mathbb{R}, \quad j = 1, \cdots, \nu, \tag{3.3.1}$$

*are satisfied. Then the vector function $\mathbf{\Phi} = (\phi_1, \cdots, \phi_\nu)^T : \mathbb{R} \to \mathbb{R}^\nu$ is refinable, with matrix refinement sequence $\{P(k)\}$, that is,*

$$\mathbf{\Phi}(x) = \sum_k P(k)\mathbf{\Phi}(2x - k), \quad x \in \mathbb{R}, \tag{3.3.2}$$

*if and only if*

$$\widehat{\mathbf{\Phi}}(\omega) := \left(\widehat{\phi}_1(\omega), \cdots, \widehat{\phi}_\nu(\omega)\right)^T = \mathcal{P}(e^{-i\omega/2})\widehat{\mathbf{\Phi}}(\omega/2), \quad \omega \in \mathbb{R}, \tag{3.3.3}$$

*for some matrix Laurent polynomial* $\mathcal{P}$*, and where, as in* (1.3.2)*,* $\mathcal{P}$ *and* $\{P(k)\}$ *are related by*

$$\mathcal{P}(z) := \frac{1}{2}\sum_k P(k)z^k. \tag{3.3.4}$$

**Proof.** Suppose first that the refinement equation (3.3.2) is satisfied for some matrix sequence $\{P(k)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$. It follows from the definitions in (2.3.5) and (3.3.3) that, for any $\omega \in \mathbb{R}$,

$$\begin{aligned}\widehat{\mathbf{\Phi}}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\Big[\sum_k P(k)\mathbf{\Phi}(2x-k)\Big]\,dx &= \frac{1}{2}\sum_k P(k)\Big[\int_{-\infty}^{\infty} e^{-i\omega(x+k)/2}\mathbf{\Phi}(x)\,dx\Big] \\ &= \Big[\frac{1}{2}\sum_k P(k)(e^{-i\omega/2})^k\Big]\,\Big[\int_{-\infty}^{\infty} e^{-i\omega(x/2)}\mathbf{\Phi}(x)\,dx\Big] \\ &= \mathcal{P}(e^{-i\omega/2})\mathbf{\Phi}(\omega/2),\end{aligned}$$

and it follows that the Fourier transform $\widehat{\mathbf{\Phi}}$ satisfies the condition (3.3.3), with the matrix Laurent polynomial $\mathcal{P}$ given by (3.3.4).

Conversely, suppose that the Fourier transform $\widehat{\mathbf{\Phi}}$ satisfies the equation (3.3.3) for some matrix Laurent polynomial $\mathcal{P}$. It then follows from (3.3.1), (3.3.4) and (3.2.27) that, for any $x \in \mathbb{R}$,

$$\begin{aligned}\mathbf{\Phi}(x) &= \left((\mathcal{F}^{-1}\widehat{\phi}_1)(x), \cdots, (\mathcal{F}^{-1}\widehat{\phi}_\nu)(x)\right)^T \\ &= \frac{1}{2\pi}\Big(\int_{-\infty}^{\infty} e^{i\omega x}\widehat{\phi}_1(\omega)\,d\omega, \cdots, \int_{-\infty}^{\infty} e^{i\omega x}\widehat{\phi}_\nu(\omega)\,d\omega\Big)^T \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left(\widehat{\phi}_1(\omega), \cdots, \widehat{\phi}_\nu(\omega)\right)^T d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\Big\{\frac{1}{2}\sum_k P(k)(e^{-i\omega/2})^k\Big\}\left(\widehat{\phi}_1(\omega/2), \cdots, \widehat{\phi}_\nu(\omega/2)\right)^T d\omega \\ &= \frac{1}{2\pi}\sum_k P(k)\frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega(x-k/2)}\left(\widehat{\phi}_1(\omega/2), \cdots, \widehat{\phi}_\nu(\omega/2)\right)^T d\omega \\ &= \frac{1}{2\pi}\sum_k P(k)\int_{-\infty}^{\infty} e^{i\omega(2x-k)}\left(\widehat{\phi}_1(\omega), \cdots, \widehat{\phi}_\nu(\omega)\right)^T d\omega \\ &= \sum_k P(k)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(2x-k)}\widehat{\phi}_1(\omega)\,d\omega, \cdots, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(2x-k)}\widehat{\phi}_\nu(\omega)\,d\omega\Big)^T \\ &= \sum_k P(k)\left((\mathcal{F}^{-1}\widehat{\phi}_1)(2x-k), \cdots, (\mathcal{F}^{-1}\widehat{\phi}_\nu)(2x-k)\right)^T,\end{aligned}$$

which, together with (3.3.1), yields

$$\mathbf{\Phi}(x) = \sum_k P(k)\Big(\phi_1(2x-k), \cdots, \phi_\nu(2x-k)\Big)^T = \sum_k P(k)\mathbf{\Phi}(2x-k),$$

and it follows that $\mathbf{\Phi}$ satisfies the refinement equation (3.3.2), with refinement sequence $\{P(k)\}$ as in (3.3.4). ■

For any integers $\nu \geq 2$ and $n \geq \rho_\nu$, with the integer $\rho_\nu$ defined by (3.1.30), we now consider the vector spline $\mathbf{\Phi}_{\nu,n}$ in Theorem 3.1.2, that is,

$$\mathbf{\Phi}_{\nu,n} := (N_n, G_{n,1}, \cdots, G_{n,\nu-1})^T, \tag{3.3.5}$$

from which it follows from Theorems 3.2.1 and 3.2.3 that

$$\widehat{\mathbf{\Phi}}_{\nu,n}(\omega) = \frac{1}{(i\omega)^{n+1}} M_{\nu,n}(e^{-i\omega})\Big(1, i\omega, \cdots, (i\omega)^{\nu-1}\Big)^T, \quad \omega \in \mathbb{R} \setminus \{0\}, \tag{3.3.6}$$

where $M_{\nu,n}$ is the $\nu \times \nu$ matrix polynomial given by

$$M_{\nu,n}(z) := \begin{bmatrix} (1-z)^{n+1} & 0 & 0 & \cdots & 0 \\ T_{n,1}(z) & 1 & 0 & \cdots & 0 \\ -T_{n,2}(z) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^\nu T_{n,\nu-1}(z) & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{3.3.7}$$

with the polynomials $\{T_{n,1}, \cdots, T_{n,\nu-1}\} \subset \pi_n$ defined by (3.2.6), (3.2.4). Note from (3.3.7) that

$$\det\big(M_{\nu,n}(z)\big) = (1-z)^{n+1} \neq 0, \quad z \in \mathbb{C} \setminus \{1\}, \tag{3.3.8}$$

according to which $M_{\nu,n}(z)$ is an invertible matrix for $z \in \mathbb{C} \setminus \{1\}$. Hence, for $\omega \in \mathbb{R} \setminus \{0\}$, we may deduce from (3.3.6) that

$$\widehat{\mathbf{\Phi}}_{\nu,n}(\omega) = \frac{1}{(i\omega)^{n+1}} M_{\nu,n}(e^{-i\omega}) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix} \begin{bmatrix} 1 \\ i\omega/2 \\ (i\omega/2)^2 \\ \vdots \\ (i\omega/2)^{\nu-1} \end{bmatrix},$$

which, together with (3.3.8), is the same as

$$\widehat{\mathbf{\Phi}}_{\nu,n}(\omega) = \frac{1}{2^{n+1}} M_{\nu,n}(e^{-i\omega}) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix} M_{\nu,n}^{-1}(e^{-i\omega/2}) \frac{1}{(i\omega/2)^{n+1}} M_{\nu,n}(e^{-i\omega/2}) \begin{bmatrix} 1 \\ i\omega/2 \\ (i\omega/2)^2 \\ \vdots \\ (i\omega/2)^{\nu-1} \end{bmatrix},$$

and thus

$$\widehat{\mathbf{\Phi}}_{\nu,n}(\omega) = \mathcal{P}_{\nu,n}(e^{-i\omega/2}) \widehat{\mathbf{\Phi}}_{\nu,n}(\omega/2), \quad \omega \in \mathbb{R} \setminus \{0\}, \tag{3.3.9}$$

where

$$\mathcal{P}_{\nu,n}(z) := \frac{1}{2^{n+1}} M_{\nu,n}(z^2) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix} M_{\nu,n}^{-1}(z). \tag{3.3.10}$$

We proceed to show that $\mathcal{P}_{\nu,n}$ is a matrix polynomial. To this end, we first note from (3.3.7) and (3.3.8) that, for any $z \in \mathbb{C} \setminus \{1\}$, we have

$$M_{\nu,n}^{-1}(z) = \frac{1}{(1-z)^{n+1}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -T_{n,1}(z) & (1-z)^{n+1} & 0 & \cdots & 0 \\ T_{n,2}(z) & 0 & (1-z)^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{\nu-1} T_{n,\nu-1}(z) & 0 & 0 & \cdots & (1-z)^{n+1} \end{bmatrix}, \tag{3.3.11}$$

which, together with (3.3.10), yields

$$\mathcal{P}_{\nu,n}(z) = \frac{1}{2^{n+1}} \begin{bmatrix} (1+z)^{n+1} & 0 & 0 & \cdots & 0 \\ \dfrac{T_{n,1}(z^2) - 2\, T_{n,1}(z)}{(1-z)^{n+1}} & 2 & 0 & \cdots & 0 \\ -\dfrac{T_{n,2}(z^2) - 2^2\, T_{n,2}(z)}{(1-z)^{n+1}} & 0 & 2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{\nu} \dfrac{T_{n,\nu-1}(z^2) - 2^{\nu-1}\, T_{n,\nu-1}(z)}{(1-z)^{n+1}} & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix}. \tag{3.3.12}$$

Let $k \in \{1, \cdots, \nu - 1\}$ be fixed, and suppose $z \in \mathbb{C}$ is such that $|z-1| < 1$. It follows from (3.2.3) and (3.2.6) that

$$(\text{Log } z)^k = T_{n,k}(z) + \sum_{j=n+1}^{\infty} t_k(j)(z-1)^j, \tag{3.3.13}$$

with the real coefficient sequence $\{t_k(j) : j = k, k+1, \cdots\}$ given by (3.2.4). Since it also holds that $|z^2 - 1| < 1$, we see from (3.3.13) that

$$(\text{Log } (z^2))^k = T_{n,k}(z^2) + \sum_{j=n+1}^{\infty} t_k(j)(z^2-1)^j. \tag{3.3.14}$$

Next, we observe that

$$(\text{Log } (z^2))^k = (2 \text{ Log } z)^k = 2^k (\text{Log } z)^k, \tag{3.3.15}$$

which, together with (3.3.13), yields

$$(\text{Log } (z^2))^k = 2^k T_{n,k}(z) + 2^k \sum_{j=n+1}^{\infty} t_k(j)(z-1)^j. \tag{3.3.16}$$

It then follows from (3.3.14) and (3.3.16) that

$$T_{n,k}(z^2) - 2^k T_{n,k}(z) = \sum_{j=n+1}^{\infty} t_k(j)\left\{2^k(z-1)^j - (z^2-1)^j\right\} = \sum_{j=n+1}^{\infty} t_k(j)(z-1)^j\left\{2^k - (z+1)^j\right\}. \tag{3.3.17}$$

For any integers $k \in \{1, \cdots, \nu - 1\}$ and $j \in \{0, 1, \cdots\}$, we have

$$2^k - (z+1)^j = 2^k - \{2 + (z-1)\}^j = 2^k - \sum_{l=0}^{\infty} \binom{j}{l} 2^{j-l}(z-1)^l = \sum_{l=0}^{\infty} \alpha_{k,j}(l)(z-1)^l, \tag{3.3.18}$$

where the real coefficient sequence $\{\alpha_{k,j}(l) : l = 0, 1, \cdots\}$ is given by

$$\alpha_{k,j}(l) := \begin{cases} 2^k - 2^j, & l = 0; \\ -\binom{j}{l} 2^{j-l}, & l = 1, 2, \cdots. \end{cases} \tag{3.3.19}$$

By substituting (3.3.18) into (3.3.17), we obtain

$$\begin{aligned} T_{n,k}(z^2) - 2^k\, T_{n,k}(z) &= \sum_{j=n+1}^{\infty} t_k(j)(z-1)^j \sum_{l=0}^{\infty} \alpha_{k,j}(l)(z-1)^l \\ &= \sum_{j=n+1}^{\infty} t_k(j) \sum_{l=0}^{\infty} \alpha_{k,j}(l)(z-1)^{l+j} \\ &= \sum_{j=n+1}^{\infty} t_k(j) \sum_{l=j}^{\infty} \alpha_{k,j}(l-j)(z-1)^l \\ &= \sum_{l=n+1}^{\infty} \left\{ \sum_{j=n+1}^{l} t_k(j)\alpha_{k,j}(l-j) \right\}(z-1)^l, \end{aligned}$$

and thus

$$T_{n,k}(z^2) - 2^k\, T_{n,k}(z) = \sum_{l=n+1}^{\infty} \beta_{n,k}(l)(z-1)^l, \tag{3.3.20}$$

where the real coefficient sequence $\{\beta_{n,k}(l) : \ l = n+1, n+2, \cdots\}$ is given by

$$\beta_{n,k}(l) := \sum_{j=n+1}^{l} t_k(j)\ \alpha_{k,j}(l-j), \quad l = n+1, n+2, \cdots . \tag{3.3.21}$$

With the notation

$$R_{n,k}(z) := T_{n,k}(z^2) - 2^k\, T_{n,k}(z), \tag{3.3.22}$$

it follows from (3.3.20) that

$$R_{n,k}(z) = \sum_{l=n+1}^{\infty} \beta_{n,k}(l)(z-1)^l, \tag{3.3.23}$$

for all $z \in \mathbb{C}$ such that $|z-1| < 1$. Since, moreover, $T_{n,k} \in \pi_n$ and (3.3.22) yield $R_{n,k} \in \pi_{2n}$, it follows from the uniqueness of infinite Taylor series that, in (3.3.23), we must have

$$\beta_{n,k}(l) = 0, \quad l = 2n+1, 2n+2, \cdots , \tag{3.3.24}$$

and thus

$$R_{n,k}(z) = \sum_{l=n+1}^{2n} \beta_{n,k}(l)(z-1)^l. \tag{3.3.25}$$

It follows from (3.3.25) that the definition

$$J_{n,k}(z) := \frac{(-1)^{k-1} R_{n,k}(z)}{2^{n+1}\ (1-z)^{n+1}} \tag{3.3.26}$$

yields the polynomial $J_{n,k} \in \pi_{n-1}$ given by

$$J_{n,k}(z) = \frac{(-1)^{n-k}}{2^{n+1}} \sum_{l=0}^{n-1} \beta_{n,k}(l+n+1)(z-1)^l. \tag{3.3.27}$$

Now observe from (3.3.27) that

$$\begin{aligned} J_{n,k}(z) &= \frac{(-1)^{n-k}}{2^{n+1}} \sum_{l=0}^{n-1} \beta_{n,k}(l+n+1)\ \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} z^m \\ &= \sum_{m=0}^{n-1} \frac{(-1)^{n-k+m}}{2^{n+1}} \Big[ \sum_{l=m}^{n-1} (-1)^l \binom{l}{m} \beta_{n,k}(l+n+1) \Big]\ z^m, \end{aligned}$$

that is,

$$J_{n,k}(z) = \sum_{m=0}^{n-1} \gamma_{n,k}(m) z^m, \tag{3.3.28}$$

where the real coefficient sequence $\{\gamma_{n,k}(m): \ m = 0, \cdots, n-1\}$ is given by

$$\gamma_{n,k}(m) := \frac{(-1)^{n-k+m}}{2^{n+1}} \sum_{l=m}^{n-1} (-1)^l \binom{l}{m} \beta_{n,k}(l+n+1), \quad m = 0, \cdots, n-1. \tag{3.3.29}$$

Based on (3.3.29). (3.3.21), (3.3.19) and (3.2.4), we can now formulate Algorithm 3.1 for the computation of the coefficient sequence $\{\gamma_{n,k}(m): \ m = 0, \cdots, n-1\}$ in (3.3.28).

*Algorithm 3.1: Explicit computation of the coefficients $\{\gamma_{n,k}(m)\}$ of $J_{n,k}$.*

Let the integers $n \in \mathbb{N}$ and $k \in \{1, \cdots, \sigma_n\}$ be given.

1. For $j = n+1, \cdots, 2n$, set

$$t_k(j) := (-1)^{k-j} \begin{cases} \dfrac{1}{j}, & k = 1; \\ \displaystyle\sum_{m=1}^{j-1} \frac{1}{m(j-m)}, & k = 2; \\ \displaystyle\sum_{m_1=1}^{j-(k-1)} \frac{1}{m_1} \sum_{m_2=1}^{j-(k-2)-m_1} \frac{1}{m_2} \cdots \\ \qquad \displaystyle\sum_{m_{k-2}=1}^{j-2-m_1-\cdots-m_{k-3}} \frac{1}{m_{k-2}} \sum_{m_{k-1}=1}^{j-1-m_1-\cdots-m_{k-2}} \frac{1}{m_{k-1}(j-m_1-\cdots-m_{k-1})}, \\ \qquad\qquad k \geq 3. \end{cases}$$

2. For $k = 1, \cdots, \sigma_n$ and $j = n+1, \cdots, 2n$, set

$$\alpha_{k,j}(l) := \begin{cases} 2^k - 2^j, & l = 0; \\ -\dbinom{j}{l} 2^{j-l}, & l \geq 1. \end{cases}$$

3. For $k = 1, \cdots, \sigma_n$ and $l = n+1, \cdots, 2n$, set

$$\beta_{n,k}(l) := \sum_{j=n+1}^{l} t_k(j) \ \alpha_{k,j}(l-j).$$

4. For $k = 1, \cdots, \sigma_n$, set

$$\gamma_{n,k}(m) := \begin{cases} \dfrac{(-1)^{n-k+m}}{2^{n+1}} \displaystyle\sum_{l=m}^{n-1} (-1)^l \binom{l}{m} \beta_{n,k}(l+n+1), & m = 0, \cdots, n-1; \\ 0, \quad m = n, n+1. \end{cases}$$

Calculating by means of Algorithm 3.1, we obtain the values in Table 3.3 for the coefficient sequences $\{\gamma_{n,k}(m): \ m = 0, \cdots, n-1\}$ in (3.3.28) for $n \in \{1, \cdots, 5\}$.

*Table 3.3: Coefficients $\{\gamma_{n,k}(m): \; m = 0, \cdots, n-1\}$ of $J_{n,k}$, for $n \in \{1, \cdots, 5\}$*

| $n$ | $k$ | $\{\gamma_{n,k}(m): \quad m = 0, \cdots, n-1\}$ |
|---|---|---|
| 1 | 1 | $\left\{\frac{1}{4}\right\}$ |
| 2 | 1 | $\left\{\frac{3}{16}, \frac{1}{16}\right\}$ |
| 3 | 1 | $\left\{\frac{11}{96}, \frac{1}{12}, \frac{1}{48}\right\}$ |
| 3 | 2 | $\left\{\frac{3}{8}, \frac{1}{4}, \frac{1}{16}\right\}$ |
| 4 | 1 | $\left\{\frac{25}{384}, \frac{29}{384}, \frac{5}{128}, \frac{1}{128}\right\}$ |
| 4 | 2 | $\left\{\frac{35}{128}, \frac{109}{384}, \frac{55}{384}, \frac{11}{384}\right\}$ |
| 4 | 3 | $\left\{\frac{35}{64}, \frac{31}{64}, \frac{15}{64}, \frac{3}{64}\right\}$ |
| 5 | 1 | $\left\{\frac{137}{3840}, \frac{37}{640}, \frac{59}{1280}, \frac{3}{160}, \frac{1}{320}\right\}$ |
| 5 | 2 | $\left\{\frac{45}{256}, \frac{97}{384}, \frac{149}{768}, \frac{5}{64}, \frac{5}{384}\right\}$ |
| 5 | 3 | $\left\{\frac{119}{256}, \frac{73}{128}, \frac{53}{128}, \frac{21}{128}, \frac{7}{256}\right\}$ |
| 5 | 4 | $\left\{\frac{45}{64}, \frac{23}{32}, \frac{31}{64}, \frac{3}{16}, \frac{1}{32}\right\}$ |

It follows from (3.3.12), together with (3.3.20), (3.3.22) and (3.3.26), that $\mathcal{P}_{\nu,n}$ is the matrix polynomial of degree $(n+1)$ given by

$$\mathcal{P}_{\nu,n}(z) = \begin{bmatrix} \left(\frac{1+z}{2}\right)^{n+1} & 0 & 0 & \cdots & 0 \\ J_{n,1}(z) & (1/2)^n & 0 & \cdots & 0 \\ J_{n,2}(z) & 0 & (1/2)^{n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ J_{n,\nu-1}(z) & 0 & 0 & \cdots & (1/2)^{n+2-\nu} \end{bmatrix}. \tag{3.3.30}$$

Also, since $\mathcal{P}_{\nu,n}$ is a matrix polynomial, and $\widehat{\mathbf{\Phi}}_{\nu,n}$ is a holomorphic function, it follows that (3.3.9) may be extended to

$$\widehat{\mathbf{\Phi}}_{\nu,n}(\omega) = \mathcal{P}_{\nu,n}(e^{-i\omega/2})\widehat{\mathbf{\Phi}}_{\nu,n}(\omega/2), \quad \omega \in \mathbb{R}. \tag{3.3.31}$$

Hence we may now apply Theorem 3.3.1, as well as (3.3.30) and (3.2.25), together with Example 1.3.2, to state the following result.

**Theorem 3.3.2** *For any integers $\nu \geq 2$ and $n \geq \rho_\nu$, with $\rho_\nu$ defined by* (3.1.30)*, let the vector spline $\mathbf{\Phi}_{\nu,n} : \mathbb{R} \to \mathbb{R}^\nu$ be given by $\mathbf{\Phi}_{\nu,n} := (N_n, G_{n,1}, \cdots, G_{n,\nu-1})^T$, with $N_n$ denoting the $n^{th}$ degree cardinal B-spline in* (1.3.3), (1.3.4)*, and where the splines*

$\{G_{n,k}: \ k=1,\cdots,\nu-1\}$ *are given as in Theorem* 3.1.1. *Then* $\mathbf{\Phi}_{\nu,n}$ *is refinable, with*

$$\mathbf{\Phi}_{\nu,n}(x) = \sum_k P_{\nu,n}(k)\mathbf{\Phi}_{\nu,n}(2x-k), \quad x \in \mathbb{R}, \tag{3.3.32}$$

*where the matrix refinement sequence* $\{P_{\nu,n}(k)\}$ *is given by* (1.3.17), *with* $\{P(k)\} = \{P_{2,1}(k)\}$, *if* $\nu = 2, \ n = 1$, *and, if* $\nu \geq 2, \ n \geq 2$, *by*

$$P_{\nu,n}(k) := 2\begin{bmatrix} \frac{1}{2^{n+1}}\binom{n+1}{k} & 0 & 0 & \cdots & 0 \\ \gamma_{n,1}(k) & (1/2)^n\delta(k) & 0 & \cdots & 0 \\ \gamma_{n,2}(k) & 0 & (1/2)^{n-1}\delta(k) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{n,\nu-1}(k) & 0 & 0 & \cdots & (1/2)^{n+2-\nu}\delta(k) \end{bmatrix}, \quad k=0,\cdots,n+1; \tag{3.3.33}$$

$$P_{\nu,n}(k) := O, \quad k \in \mathbb{Z}\setminus\{0,\cdots,n+1\}, \tag{3.3.34}$$

*with the sequences* $\{\gamma_{n,j}(k): \ k=0,\cdots,n-1\}, \ j=1,\cdots,\nu-1$, *obtained as in Algorithm* 3.1, *together with*

$$\gamma_{n,j}(k) := 0, \quad k=\nu,\cdots,n+1; \quad j=1,\cdots,\nu-1. \tag{3.3.35}$$

Calculating by means of Algorithm 3.1 and (3.3.35), we now obtain the following matrix refinement sequences $\{P_{\nu,n}(k): \ k=0,\cdots,n+1\}$ for $\nu$ and $n$ as shown.

*Table 3.4: Matrix refinement sequence* $\{P_{\nu,n}(k): \ k=0,\cdots,n+1\}$

| $\nu$ | $n$ | $\{P_{\nu,n}(k): \ k=0,\cdots,n+1\}$ |
|---|---|---|
| 2 | 1 | $P_{2,1}(0) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}, \quad P_{2,1}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{2,1}(2) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}.$ |
| 2 | 2 | $P_{2,2}(0) = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{1}{2} \end{bmatrix}, \quad P_{2,2}(1) = \begin{bmatrix} \frac{3}{4} & 0 \\ \frac{1}{8} & 0 \end{bmatrix}, \quad P_{2,2}(2) = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{2,2}(3) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix}.$ |
| 2 | 3 | $P_{2,3}(0) = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{11}{48} & \frac{1}{4} \end{bmatrix}, \quad P_{2,3}(1) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{6} & 0 \end{bmatrix}, \quad P_{2,3}(2) = \begin{bmatrix} \frac{3}{4} & 0 \\ \frac{1}{24} & 0 \end{bmatrix}, \quad P_{2,3}(3) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix},$ $P_{2,3}(4) = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & 0 \end{bmatrix}.$ |

| $\nu$ | $n$ | $\{P_{\nu,n}(k): \; k = 0, \cdots, n+1\}$ |
|---|---|---|
| 3 | 3 | $P_{3,3}(0) = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ \frac{11}{48} & \frac{1}{4} & 0 \\ \frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad P_{3,3}(1) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \\ -\frac{85}{72} & 0 & 0 \end{bmatrix}, \quad P_{3,3}(2) = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{24} & 0 & 0 \\ \frac{43}{72} & 0 & 0 \end{bmatrix},$ $P_{3,3}(3) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{3,3}(4) = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$ |
| 3 | 4 | $P_{3,4}(0) = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ \frac{25}{192} & \frac{1}{8} & 0 \\ \frac{203}{160} & 0 & \frac{1}{4} \end{bmatrix}, \quad P_{3,4}(1) = \begin{bmatrix} \frac{5}{16} & 0 & 0 \\ \frac{29}{192} & 0 & 0 \\ -\frac{719}{480} & 0 & 0 \end{bmatrix}, \quad P_{3,4}(2) = \begin{bmatrix} \frac{5}{8} & 0 & 0 \\ \frac{5}{64} & 0 & 0 \\ \frac{701}{480} & 0 & 0 \end{bmatrix},$ $P_{3,4}(3) = \begin{bmatrix} \frac{5}{8} & 0 & 0 \\ \frac{1}{64} & 0 & 0 \\ -\frac{307}{960} & 0 & 0 \end{bmatrix}, \quad P_{3,4}(4) = \begin{bmatrix} \frac{5}{16} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{3,4}(5) = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$ |
| 3 | 5 | $P_{3,5}(0) = \begin{bmatrix} \frac{1}{32} & 0 & 0 \\ \frac{137}{1920} & \frac{1}{16} & 0 \\ \frac{207}{224} & 0 & \frac{1}{8} \end{bmatrix}, \quad P_{3,5}(1) = \begin{bmatrix} \frac{3}{16} & 0 & 0 \\ \frac{37}{320} & 0 & 0 \\ -\frac{1957}{1344} & 0 & 0 \end{bmatrix}, \quad P_{3,5}(2) = \begin{bmatrix} \frac{15}{32} & 0 & 0 \\ \frac{59}{640} & 0 & 0 \\ \frac{4673}{2240} & 0 & 0 \end{bmatrix},$ $P_{3,5}(3) = \begin{bmatrix} \frac{5}{8} & 0 & 0 \\ \frac{3}{80} & 0 & 0 \\ -\frac{10267}{11200} & 0 & 0 \end{bmatrix}, \quad P_{3,5}(4) = \begin{bmatrix} \frac{15}{32} & 0 & 0 \\ \frac{1}{160} & 0 & 0 \\ \frac{7351}{33600} & 0 & 0 \end{bmatrix}, \quad P_{3,5}(5) = \begin{bmatrix} \frac{3}{16} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$ $P_{3,5}(5) = \begin{bmatrix} \frac{1}{32} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$ |

**Remark 3.3.1** Observe from Table 3.4 that the matrix refinement sequence $\{P_{2,1}(k)\}$ corresponds precisely with (1.3.17), as it to be expected by virtue of the final paragraph in Section 3.1.

## 3.4 Integer-shift linear independence and $l^2$-stability

In this section, we investigate the integer-shift linear independence and $l^2$-stability of the refinable vector spline $\mathbf{\Phi}_{\nu,n}$ of Theorems 3.1.2 and 3.3.2. After first recalling that such matrix linear independence and stability have already been established for the case $\nu = 2$,

$n = 1$ in Example 1.4.2, we next let $n \geq \nu \geq 2$, and proceed to apply the equivalent Fourier transform formulations in Theorems 2.3.3 and 2.3.4 of, respectively, integer-shift linear independence and $l^2$-stability. Note that the applicability of Theorems 2.3.3 and 2.3.4 is guaranteed by the fact that $\mathbf{\Phi}_{\nu,n} \in \mathbf{C}_0(\mathbb{R})$ for $n \geq \nu \geq 2$, as follows from (3.1.32).

As in (2.3.9) of Theorem 2.3.3, for any fixed $\omega \in \mathbb{C}$, we let $\{c_0, c_1, \cdots, c_{\nu-1}\}$ denote any sequence in $\mathbb{C}$ satisfying the condition

$$c_0\widehat{N}_n(\omega + 2\pi k) + \sum_{j=1}^{\nu-1} c_j\widehat{G}_{n,j}(\omega + 2\pi k) = 0, \quad k \in \mathbb{Z}. \tag{3.4.1}$$

Suppose first

$$\omega \notin \{2\pi l \ : \ l \in \mathbb{Z}\}, \tag{3.4.2}$$

according to which then

$$\omega + 2\pi k \neq 0, \quad k \in \mathbb{Z}. \tag{3.4.3}$$

It then follows from (3.2.1) and (3.2.5) in, respectively, Theorems 3.2.1 and 3.2.3, that the condition (3.4.1) is equivalent to

$$c_0\Big(\frac{1 - e^{-i\omega}}{i(\omega + 2\pi k)}\Big)^{n+1} + \sum_{j=1}^{\nu-1} c_j(-1)^j \frac{\big(-i(\omega + 2\pi k)\big)^j - T_{n,j}(e^{-i\omega})}{\big(i(\omega + 2\pi k)\big)^{n+1}} = 0, \quad k \in \mathbb{Z}, \tag{3.4.4}$$

and thus, by using also (3.4.3),

$$c_0(1 - e^{-i\omega})^{n+1} + \sum_{j=1}^{\nu-1} c_j(-1)^j\Big\{\big(-i(\omega + 2\pi k)\big)^j - T_{n,j}(e^{-i\omega})\Big\} = 0, \quad k \in \mathbb{Z}. \tag{3.4.5}$$

By setting $k = 0$ in (3.4.5), we obtain

$$c_0(1 - e^{-i\omega})^{n+1} + \sum_{j=1}^{\nu-1} c_j(-1)^j\Big\{(-i\omega)^j - T_{n,j}(e^{-i\omega})\Big\} = 0. \tag{3.4.6}$$

Hence we may subtract (3.4.6) from (3.4.5) to deduce that

$$\sum_{j=1}^{\nu-1} c_j\Big\{\big(i(\omega + 2\pi k)\big)^j - (i\omega)^j\Big\} = 0, \quad k \in \mathbb{Z}. \tag{3.4.7}$$

Now observe that the definition

$$F(z) := \sum_{j=1}^{\nu-1} c_j\Big\{\big(i(\omega + 2\pi z)\big)^j - (i\omega)^j\Big\}, \quad z \in \mathbb{C}, \tag{3.4.8}$$

yields a polynomial $F$ with complex coefficients and of degree at most $\nu - 1$. Moreover, (3.4.8) and (3.4.7) imply that $F(k) = 0$, $k \in \mathbb{Z}$, according to which $F$ must be the zero polynomial, that is,

$$\sum_{j=1}^{\nu-1} c_j \Big\{ \big(i(\omega + 2\pi z)\big)^j - (i\omega)^j \Big\} = 0, \quad z \in \mathbb{C}, \tag{3.4.9}$$

or equivalently,

$$\sum_{j=1}^{\nu-1} c_j \big\{ (z+\zeta)^j - \zeta^j \big\} = 0, \quad z \in \mathbb{C}, \tag{3.4.10}$$

where

$$\zeta := i\omega. \tag{3.4.11}$$

It follows from (3.4.10) that, for any $z \in \mathbb{C}$,

$$0 = \sum_{j=1}^{\nu-1} c_j \left[ \sum_{l=0}^{j} \binom{j}{l} \zeta^{j-l} z^l - \zeta^j \right] = \sum_{j=1}^{\nu-1} c_j \left[ \sum_{l=1}^{j} \binom{j}{l} \zeta^{j-l} z^l \right] = \sum_{l=1}^{\nu-1} \left[ \sum_{j=l}^{\nu-1} \binom{j}{l} \zeta^{j-l} c_j \right] z^l,$$

that is,

$$\sum_{l=1}^{\nu-1} \left\{ \sum_{j=l}^{\nu-1} \binom{j}{l} \zeta^{j-l} c_j \right\} z^l = 0, \quad z \in \mathbb{C}, \tag{3.4.12}$$

and thus

$$\sum_{j=l}^{\nu-1} \binom{j}{l} \zeta^{j-l} c_j = 0, \quad l = 1, \cdots, \nu - 1. \tag{3.4.13}$$

By setting $l = \nu - 1$ in (3.4.13), we obtain

$$c_{\nu-1} = 0. \tag{3.4.14}$$

If $\nu \geq 3$, we see from (3.4.13) that

$$c_l = - \sum_{j=l+1}^{\nu-1} \binom{j}{l} \zeta^{j-l} c_j, \quad l = 1, \cdots, \nu - 2,$$

according to which, it follows inductively from (3.4.14) that

$$c_1 = \cdots = c_{\nu-2} = 0. \tag{3.4.15}$$

Hence we may now substitute (3.4.14) and (3.4.15) into (3.4.5) to obtain

$$c_0 (1 - e^{-i\omega})^{n+1} = 0. \tag{3.4.16}$$

But (3.4.2) implies $1 - e^{-i\omega} \neq 0$, so that we may deduce from (3.4.16) that $c_0 = 0$. Hence we have now shown that, for any $\omega \in \mathbb{C}$ such that (3.4.2) holds, the condition (3.4.1) implies $c_0 = \cdots = c_{\nu-1} = 0$.

Suppose next that $\omega = 2\pi l$ for some $l \in \mathbb{Z}$. It then follows from (3.2.1) and (3.2.5), together with the fact that (3.2.6) gives $T_{n,j}(1) = 0$ for $j \in \{1, \cdots, \nu - 1\}$, that the condition (3.4.1) has the equivalent formulation

$$\left.\begin{aligned} &\sum_{j=1}^{\nu-1} c_j (2\pi i)^j (l+k)^j = 0, \quad k \in \mathbb{Z} \setminus \{-l\}; \\ &c_0 + \sum_{j=1}^{\nu-1} c_j \int_0^{\nu} G_{n,j}(x)dx = 0, \quad k = -l. \end{aligned}\right\} \tag{3.4.17}$$

Let

$$\widetilde{F}(z) := \sum_{j=1}^{\nu-1} c_j (2\pi i)^j (l+z)^j, \quad z \in \mathbb{C}, \tag{3.4.18}$$

according to which $\widetilde{F}$ is a polynomial with complex coefficients and of degree at most $\nu - 1$. Since, moreover, (3.4.18) and (3.4.17) imply $\widetilde{F}(k) = 0, \quad k \in \mathbb{Z} \setminus \{-l\}$, we deduce that $\widetilde{F}$ must be the zero polynomial, that is,

$$\sum_{j=1}^{\nu-1} c_j (2\pi i)^j (l+z)^j = 0, \quad z \in \mathbb{C}, \tag{3.4.19}$$

or equivalently,

$$\sum_{j=1}^{\nu-1} c_j (2\pi i)^j z^j = 0, \quad z \in \mathbb{C}. \tag{3.4.20}$$

It then follows from (3.4.20) that

$$c_1 = \cdots = c_{\nu-1} = 0. \tag{3.4.21}$$

By substituting (3.4.21) into the second line of (3.4.17), we obtain $c_0 = 0$. Hence, if $\omega = 2\pi l$ for some $l \in \mathbb{Z}$, the condition (3.4.1) implies $c_0 = \cdots = c_{\nu-1} = 0$.

We have therefore now established the fact that, for any $\omega \in \mathbb{C}$, the only sequence $\{c_0, \cdots, c_{\nu-1}\}$ in $\mathbb{C}$ satisfying the identity (3.4.1) is the zero sequence, that is, $c_0 = \cdots = c_{\nu-1} = 0$. Hence we may now apply Theorems 2.3.3 and 2.3.4, as well as Example 1.4.2, to deduce the following integer-shift linear independence and $l^2$-stability result for the refinable vector spline $\mathbf{\Phi}_{\nu,n}$ of this chapter.

**Theorem 3.4.1** *For any integers $\nu \geq 2$ and $n \geq \rho_\nu$, with $\rho_\nu$ defined by* (3.1.30)*, the refinable vector spline $\mathbf{\Phi}_{\nu,n}$ of Theorems* 3.1.2 *and* 3.3.2 *possesses matrix linearly independent integer shifts, as well as $l^2$-stable integer shifts, on $\mathbb{R}$.*

In this chapter, by extending the construction introduced in [31], we derived explicit formulations of a class of arbitrarily smooth refinable vector splines, and also produced algorithms to compute their corresponding matrix refinement sequences. Moreover, it has been proven that these vector splines also possess integer-shift linear independence, as well as integer-shift stability, on $\mathbb{R}$, by using their Fourier transform formulations given in Section 3.2. In Chapter 4, we proceed to investigate a third class of interpolatory refinable vector splines with desirable properties, and with support $[-1, 1]$.

# Chapter 4

# HERMITE REFINABLE VECTOR SPLINES

For any integer $\nu \in \mathbb{N}$, we present in this chapter a construction, as based on Hermite polynomial interpolation, of a refinable vector spline

$$\mathbf{\Phi} = \mathbf{\Phi}_{\nu}^{H} = \left(\phi_{\nu,1}^{H}, \cdots, \phi_{\nu,\nu}^{H}\right)^{T} : \mathbb{R} \to \mathbb{R}^{\nu}.$$

## 4.1 Construction and refinability

Our basic building blocks will, for any $\nu \in \mathbb{N}$, be the Hermite interpolation polynomials $\{f_{\nu,k} : k = 1, \cdots, \nu\} \subset \pi_{2\nu-1}$, as uniquely determined by the conditions

$$\left.\begin{aligned} &f_{\nu,k}^{(l)}(0) = \delta(l+1-k); \\ &f_{\nu,k}^{(l)}(1) = 0, \end{aligned}\right\} \quad l = 0, \cdots, \nu - 1, \tag{4.1.1}$$

and for which the existence and uniqueness in $\pi_{2\nu-1}$ is guaranteed by a standard result (see e.g. [38]).

Observe that, for $\nu = 1$, we have

$$f_{1,1}(x) = 1 - x. \tag{4.1.2}$$

The following result is an immediate consequence of (4.1.1).

**Theorem 4.1.1** *For any integer $\nu$, let the functions $\left\{\phi_{\nu,k}^{H} : \mathbb{R} \to \mathbb{R};\ k = 1, \cdots, \nu\right\}$ be defined by*

$$\phi_{\nu,k}^{H}(x) := \left\{\begin{array}{l} f_{\nu,k}(x), \quad x \in [0,1]; \\ (-1)^{k-1} f_{\nu,k}(-x), \quad x \in [-1,0); \\ 0, \quad x \in \mathbb{R} \setminus [-1,1], \end{array}\right\} \quad k = 1, \cdots, \nu, \tag{4.1.3}$$

*with* $\{f_{\nu,k} : \ k = 1,\cdots,\nu\}$ *denoting the Hermite interpolation polynomials in* $\pi_{2\nu-1}$*, as uniquely determined by the conditions* (4.1.1)*. Then:*

(a) *For each* $k \in \{1,\cdots,\nu\}$*, the function* $\phi^H_{\nu,k}$ *belongs to the spline space* $S_{2\nu-1,\nu-1}(\mathbb{Z})$*, as defined according to* (3.1.2)*, that is,* $\phi^H_{\nu,k}$ *is a spline of degree* $2\nu-1$*, and with*

$$\phi^H_{\nu,k} \in C^{\nu-1}(\mathbb{R}). \tag{4.1.4}$$

(b) *For each* $k \in \{1,\cdots,\nu\}$*, the function* $\phi^H_{\nu,k}$ *is compactly supported, with*

$$\text{supp } \ \phi^H_{\nu,k} = [-1,1]. \tag{4.1.5}$$

(c) *For each* $k \in \{1,\cdots,\nu\}$*, the function* $\phi^H_{\nu,k}$ *satisfies the Hermite interpolation conditions*

$$\left(\phi^H_{\nu,k}\right)^{(l)}(j) = \delta(l+1-k)\delta(j), \quad l = 0,\cdots,\nu-1; \quad k = 1,\cdots,\nu; \quad j \in \mathbb{Z}. \tag{4.1.6}$$

(d) *For any sufficiently differentiable function* $F : \ \mathbb{R} \to \mathbb{R}$*, the spline* $s \in S_{2\nu-1,\nu-1}(\mathbb{Z})$ *defined by*

$$s(x) := \sum_j \sum_{k=1}^{\nu} F^{(k-1)}(j)\phi^H_{\nu,k}(x-j) \tag{4.1.7}$$

*satisfies the Hermite interpolation conditions*

$$s^{(l)}(\mu) = F^{(l)}(\mu), \quad l = 0,\cdots,\nu-1; \quad \mu \in \mathbb{Z}. \tag{4.1.8}$$

We proceed to prove that the vector spline with component splines given in Theorem 4.1.1, is in fact a refinable vector function, as follows.

**Theorem 4.1.2** *For any* $\nu \in \mathbb{N}$*, the Hermite interpolation vector spline* $\mathbf{\Phi}^H_\nu = \left(\phi^H_{\nu,1},\cdots,\phi^H_{\nu,\nu}\right)^T : \ \mathbb{R} \to \mathbb{R}^\nu$*, as defined according to Theorem* 4.1.1*, is refinable, with*

$$\mathbf{\Phi}^H_\nu(x) = \sum_k P^H_\nu(k)\mathbf{\Phi}^H_\nu(2x-k), \quad x \in \mathbb{R}, \tag{4.1.9}$$

*and where the matrix refinement sequence* $\left\{P^H_\nu(k)\right\}$ *is given by*

$$\left[P^H_\nu(k)\right]_{mn} = \frac{1}{2^{n-1}}\left(\phi^H_{\nu,m}\right)^{(n-1)}(k/2), \quad m,n = 1,\cdots,\nu; \quad k \in \{-1,0,1\}; \tag{4.1.10}$$

$$P^H_\nu(k) = O, \quad k \in \mathbb{Z}\setminus\{-1,0,1\}. \tag{4.1.11}$$

**Proof.** First, observe from (4.1.5) that both sides of (4.1.9) are equal to $\mathbf{0}$ for $x \in \mathbb{R} \setminus (-1,1)$. Hence, it will suffice to prove (4.1.9), (4.1.10), (4.1.11) for $x \in (-1,1)$.

To this end, we define the vector spline

$$\mathbf{F}_\nu(x) = \Big(F_{\nu,1}, \cdots, F_{\nu,\nu}\Big)^T := \sum_{k=-1}^{1} P_\nu^H(k)\mathbf{\Phi}_\nu^H(2x-k), \tag{4.1.12}$$

according to which our proof will be complete if we can show that

$$F_{\nu,m}(x) = \phi_{\nu,m}^H(x), \quad x \in (-1,1), \quad m = 1, \cdots, \nu. \tag{4.1.13}$$

To prove (4.1.13), we fix $m \in \{1, \cdots, \nu\}$, and apply (4.1.13), (4.1.12) and (4.1.10) to obtain

$$F_{\nu,m}^{(l)}(x) = \sum_{k=-1}^{1}\sum_{n=1}^{\nu} \frac{1}{2^{n-1-l}} \big(\phi_{\nu,m}^H\big)^{(n-1)}(k/2)\big(\phi_{\nu,n}^H\big)^{(l)}(2x-k), \quad l = 0, \cdots, \nu-1,$$

and thus

$$F_{\nu,m}^{(l)}(\mu/2) = \sum_{k=-1}^{1}\sum_{n=1}^{\nu} \frac{1}{2^{n-1-l}} \big(\phi_{\nu,m}^H\big)^{(n-1)}(k/2)\big(\phi_{\nu,n}^H\big)^{(l)}(\mu-k), \quad l = 0, \cdots, \nu-1, \quad \mu \in \mathbb{R}. \tag{4.1.14}$$

By applying (4.1.6) in (4.1.14), we obtain

$$F_{\nu,m}^{(l)}(\mu/2) = \sum_{n=1}^{\nu} \frac{1}{2^{n-1-l}} \big(\phi_{\nu,m}^H\big)^{(n-1)}(\mu/2)\delta(l+1-n) = \big(\phi_{\nu,n}^H\big)^{(l)}(\mu/2), \tag{4.1.15}$$

that is,

$$F_{\nu,m}^{(l)}(\mu/2) = \big(\phi_{\nu,n}^H\big)^{(l)}(\mu/2), \quad l = 0, \cdots, \nu-1, \quad \mu \in \mathbb{Z}. \tag{4.1.16}$$

Let the integer $m \in \{1, \cdots, \nu\}$ be fixed. It follows from (4.1.16) that the spline

$$G_{\nu,m} := F_{\nu,m} - \phi_{\nu,m}^H \tag{4.1.17}$$

satisfies the conditions

$$G_{\nu,m}^{(l)}(\mu/2) = 0, \quad l = 0, \cdots, \nu-1, \quad \mu \in \mathbb{Z}. \tag{4.1.18}$$

Now let $x \in (-1,1)$ and denote by $k$ the (unique) integer in the set $\{-2,-1,0,1\}$ for which it holds that $x \in \Big[\frac{k}{2}, \frac{k+1}{2}\Big)$.

From the definitions (4.1.17), (4.1.12), and since $\phi_{\nu,m}^H$ is a polynomial spline of degree $2\nu-1$ with knots (or breakpoints) at the integers, we deduce that there is a polynomial $H_{\nu,m,k} \in \pi_{2\nu-1}$, such that

$$G_{\nu,m}(x) = H_{\nu,m,k}(x), \quad x \in \left[\frac{k}{2}, \frac{k+1}{2}\right), \tag{4.1.19}$$

and for which it then follows from (4.1.18) that

$$H_{\nu,m,k}^{(l)}(k/2) = H_{\nu,m,k}^{(l)}\big((k+1)/2\big) = 0, \quad l = 0, \cdots, \nu - 1. \tag{4.1.20}$$

According to (4.1.20), we have

$$H_{\nu,m,k}(x) = \left(x - \frac{k}{2}\right)^{\nu}\left(x - \frac{k+1}{2}\right)^{\nu}\widetilde{H}_{\nu,m,k}(x), \tag{4.1.21}$$

for some polynomial $\widetilde{H}_{\nu,m,k}$. Since $H_{\nu,m,k} \in \pi_{2\nu-1}$, we deduce that the polynomial $\widetilde{H}_{\nu,m,k}$ in (4.1.21) must be the zero polynomial, and hence $H_{\nu,m,k}$ is also the zero polynomial. By recalling also the argument leading to (4.1.19), we have therefore now shown that

$$G_{\nu,m}(x) = 0, \quad x \in (-1, 1),$$

so that the desired result (4.1.13) then follows from the definition (4.1.17). ∎

## 4.2 Explicit and recursive formulations

We proceed in this section to firstly derive an explicit formulation for the Hermite interpolation vector spline of Theorem 4.1.2.

First, observe from (4.1.3) and (4.1.2) that the case $\nu = 1$ is given by

$$\mathbf{\Phi}_1^H = \phi_{1,1}^H = N_1(\cdot - 1), \tag{4.2.1}$$

where $N_1$ is the shifted hat function given by (1.3.6), so that $\mathbf{\Phi}_1^H = \phi_{1,1}^H$ is the linear hat function supported on $[-1, 1]$.

Suppose next $\nu \geq 2$. It then follows from (4.1.1) with $k = \nu$, together with $f_{\nu,\nu} \in \pi_{2\nu-1}$, that

$$f_{\nu,\nu}(x) = \frac{1}{(\nu-1)!}x^{\nu-1}(1-x)^{\nu}. \tag{4.2.2}$$

Observe from (4.1.2) that (4.2.2) actually also holds for $\nu = 1$.

Now let $k \in \{1, \cdots, \nu - 1\}$ be fixed. We shall rely on the polynomial $g_{\nu,k} \in \pi_{2\nu-1}$ defined by

$$g_{\nu,k}(x) := f_{\nu,k}(x) + (-1)^{k+1} f_{\nu,k}(1-x), \tag{4.2.3}$$

according to which

$$g_{\nu,k}^{(l)}(x) := f_{\nu,k}^{(l)}(x) + (-1)^{k+1+l} f_{\nu,k}^{(l)}(1-x), \quad l = 0, \cdots, \nu - 1, \tag{4.2.4}$$

and thus, by applying also (4.1.1),

$$g_{\nu,k}^{(l)}(0) = g_{\nu,k}^{(l)}(1) = \delta(l+1-k), \quad l = 0, \cdots, \nu-1. \tag{4.2.5}$$

Since $g_{\nu,k}^{(k-1)} \in \pi_{2\nu-k}$, it follows from (4.2.5) that

$$g_{\nu,k}^{(k-1)}(x) = 1 + x^{\nu-k+1}u_{\nu,k}(x) \tag{4.2.6}$$

for some polynomial $u_{\nu,k} \in \pi_{\nu-1}$. For $k = 1$, we note from (4.2.5), together with the uniqueness in $\pi_{2\nu-1}$ of the Hermite polynomial interpolant with respect to precisely $2\nu$ interpolation conditions, that $u_{\nu,1}$ is the zero polynomial, that is,

$$g_{\nu,1}(x) = 1, \quad x \in \mathbb{R}. \tag{4.2.7}$$

For $k \in \{2, \cdots, \nu-1\}$ ( if $\nu \geq 3$), we deduce from (4.2.5) and (4.2.6) that

$$g_{\nu,k}(x) = \begin{cases} \int_0^x \left\{1 + t^{\nu-1}u_{\nu,k}(t)\right\} dt, & k = 2; \\ \int_0^x \int_0^{t_{k-1}} \cdots \int_0^{t_2} \left\{1 + t_1^{\nu-k+1}u_{\nu,k}(t_1)\right\} dt_1 \cdots dt_{k-1}, & k = 3, \cdots, \nu-1 \text{ ( if } \nu \geq 4), \end{cases} \tag{4.2.8}$$

and thus

$$g_{\nu,k}(x) = \frac{x^{k-1}}{(k-1)!} + x^{\nu}v_{\nu,k}(x), \quad k = 2, \cdots, \nu-1, \tag{4.2.9}$$

for some polynomial $v_{\nu,k} \in \pi_{\nu-1}$, with, from (4.2.7),

$$v_{\nu,1} = \text{ the zero polynomial.} \tag{4.2.10}$$

Now observe from $f \in \pi_{2\nu-1}$ and (4.1.1) that

$$f_{\nu,k}(x) = x^{k-1}(1-x)^{\nu}h_{\nu,k}(x) \tag{4.2.11}$$

for some polynomial $h_{\nu,k} \in \pi_{\nu-k}$. By substituting (4.2.11) into (4.2.3), and using (4.2.9), we obtain the identity

$$(1-x)^{\nu}h_{\nu,k}(x) + (-1)^{k+1}(1-x)^{k-1}x^{\nu-k+1}h_{\nu,k}(1-x) = \frac{1}{(k-1)!} + x^{\nu-k+1}v_{\nu,k}(x), \quad x \in \mathbb{R}, \tag{4.2.12}$$

according to which

$$(1-x)^{\nu}h_{\nu,k}(x) = \frac{1}{(k-1)!} + x^{\nu-k+1}w_{\nu,k}(x), \quad x \in \mathbb{R}, \tag{4.2.13}$$

for some polynomial $w_{\nu,k} \in \pi_{\nu-1}$. Let $x \in (-1,1)$. It then follows from (4.2.13) that

$$h_{\nu,k}(x) = \frac{1}{(1-x)^{\nu}}\Big\{\frac{1}{(k-1)!} + x^{\nu-k+1}w_{\nu,k}(x)\Big\}. \tag{4.2.14}$$

By differentiating the (absolutely convergent) geometric series

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j \tag{4.2.15}$$

$(\nu-1)$ times, we obtain

$$\frac{1}{(1-x)^{\nu}} = \sum_{j=0}^{\infty}\binom{\nu-1+j}{j}x^j. \tag{4.2.16}$$

It follows from (4.2.14) and (4.2.16) that

$$h_{\nu,k}(x) = \frac{1}{(k-1)!}\sum_{j=0}^{\nu-k}\binom{\nu-1+j}{j}x^j + x^{\nu-k+1}\sum_{j=0}^{\infty}\beta_{\nu,k}(j)x^j, \tag{4.2.17}$$

for some real coefficient sequence $\{\beta_{\nu,k}(j) : \ j = 0,1,\cdots\}$. Since, moreover, $h_{\nu,k} \in \pi_{\nu-k}$, it follows from (4.2.17), together with the uniqueness of infinite Taylor series, that $\beta_{\nu,k}(j) = 0, \ j = 0,1,\cdots$, in (4.2.17), and thus

$$h_{\nu,k}(x) = \frac{1}{(k-1)!}\sum_{j=0}^{\nu-k}\binom{\nu-1+j}{j}x^j. \tag{4.2.18}$$

Hence, we may now combine (4.2.11), (4.2.18), (4.2.2) and (4.1.2) to state the following explicit formulation.

**Theorem 4.2.1** *For any integers $\nu \in \mathbb{N}$ and $k \in \{1,\cdots,\nu\}$, the polynomial $f_{\nu,k} \in \pi_{2\nu-1}$ in the definition* (4.1.3) *of the spline $\phi^H_{\nu,k}$ is given by the formula*

$$f_{\nu,k}(x) = \frac{1}{(k-1)!}x^{k-1}\ (1-x)^{\nu}\sum_{j=0}^{\nu-k}\binom{\nu-1+j}{j}x^j. \tag{4.2.19}$$

Next, we derive a computationally efficient recursive formulation of the polynomial $f_{\nu,k}$ with respect to the index $\nu$. Let the integers $\nu \in \mathbb{N}$ and $k \in \{1,\cdots,\nu\}$ be fixed. We have already shown that the polynomial $h_{\nu,k} \in \pi_{\nu-k}$ in the formulation (4.2.11) satisfies the identity (4.2.13) for some polynomial $w_{\nu,k} \in \pi_{\nu-1}$ if $\nu \geq 2$ and $k \in \{1,\cdots,\nu-1\}$. We claim that (4.2.13) is in fact satisfied for all integers $\nu \in \mathbb{N}$ and $k \in \{1,\cdots,\nu\}$. To prove this statement, we first observe from (4.2.2) and (4.2.19) in Theorem 4.2.1 that

$$h_{\nu,\nu}(x) = \frac{1}{(\nu-1)!}, \quad x \in \mathbb{R}, \tag{4.2.20}$$

and thus

$$(1-x)^\nu h_{\nu,\nu}(x) = \frac{1}{(\nu-1)!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-1)^j x^j = \frac{1}{(\nu-1)!} + \frac{x}{(\nu-1)!}\sum_{j=1}^{\nu}(-1)^j\binom{\nu}{j}x^{j-1},$$

which proves (4.2.13) for $k=\nu$, with

$$w_{\nu,\nu}(x) = \frac{1}{(\nu-1)!}\sum_{j=0}^{\nu-1}(-1)^{j+1}\binom{\nu}{j+1}x^j, \tag{4.2.21}$$

that is, $w_{\nu,\nu} \in \pi_{\nu-1}$.

For $\nu = 1$ and $k = 1$, it follows from (4.2.11) and (4.1.2) that

$$(1-x)h_{1,1}(x) = 1-x,$$

which yields (4.2.13) , with $w_{1,1}(x) = -1$, for any $x \in \mathbb{R}$, that is, $w_{1,1} \in \pi_0$. Hence we have now proven the validity of the identity (4.2.13) for all integers $\nu \in \mathbb{N}$ and $k \in \{1, \cdots, \nu\}$.

For any fixed $\nu \in \mathbb{N}$ and $k \in \{1, \cdots, \nu\}$, we may replace $\nu$ by $\nu+1$ in (4.2.13) to obtain

$$(1-x)^{\nu+1}h_{\nu+1,k}(x) = \frac{1}{(k-1)!} + x^{\nu-k+2}w_{\nu+1,k}(x), \quad x \in \mathbb{R}, \tag{4.2.22}$$

for some polynomial $w_{\nu+1,k} \in \pi_\nu$. By subtracting (4.2.13) from (4.2.22), we deduce the identity

$$(1-x)^\nu\Big\{(1-x)h_{\nu+1,k}(x) - h_{\nu,k}(x)\Big\} = x^{\nu-k+1}\Big\{x\ w_{\nu+1,k}(x) - w_{\nu,k}(x)\Big\}, \quad x \in \mathbb{R}. \tag{4.2.23}$$

It then follows from (4.2.23) that

$$(1-x)h_{\nu+1,k}(x) - h_{\nu,k}(x) = x^{\nu-k+1}\widetilde{h}_{\nu,k}(x), \tag{4.2.24}$$

for some polynomial $\widetilde{h}_{\nu,k}$. Since, moreover, $h_{\nu,k} \in \pi_{\nu-k}$, we deduce from (4.2.24) that $\widetilde{h}_{\nu,k}$ is a linear polynomial, according to which

$$(1-x)h_{\nu+1,k}(x) - h_{\nu,k}(x) = x^{\nu-k+1}(Ax+B), \tag{4.2.25}$$

for some constants $A$ and $B$. By setting $x=1$ in (4.2.25), we obtain

$$A + B = -h_{\nu,k}(1). \tag{4.2.26}$$

Now apply the formula (4.2.18), to obtain

$$
\begin{aligned}
&(k-1)!h_{\nu,k}(1)\\
&=\sum_{j=0}^{\nu-k}\binom{\nu-1+j}{j}\\
&=\sum_{j=0}^{\nu-k}\left\{\binom{\nu+j}{j}-\binom{\nu-1+j}{j-1}\right\}\\
&=\{1-0\}+\left\{\binom{\nu+1}{1}-1\right\}+\left\{\binom{\nu+2}{2}-\binom{\nu+1}{1}\right\}+\cdots+\left\{\binom{2\nu-k}{\nu-k}-\binom{2\nu-k-1}{\nu-k-1}\right\}\\
&=\binom{2\nu-k}{\nu-k},
\end{aligned}
$$

and thus

$$h_{\nu,k}(1)=\frac{1}{(k-1)!}\binom{2\nu-k}{\nu-k}. \tag{4.2.27}$$

It then follows from (4.2.26) and (4.2.27) that the identity (4.2.25) is given by

$$(1-x)h_{\nu+1,k}(x)-h_{\nu,k}(x)=x^{\nu-k+1}\left\{Ax-\left(A+\frac{1}{(k-1)!}\binom{2\nu-k}{\nu-k}\right)\right\}. \tag{4.2.28}$$

By using also (4.2.18), we now equate the leading coefficients in the left and right-hand sides of (4.2.28), to get

$$A=-\frac{1}{(k-1)!}\binom{2\nu-k+1}{\nu-k+1}. \tag{4.2.29}$$

Note from (4.2.29) that

$$A+\frac{1}{(k-1)!}\binom{2\nu-k}{\nu-k}=-\frac{1}{(k-1)!}\left\{\binom{2\nu-k+1}{\nu-k+1}-\binom{2\nu-k}{\nu-k}\right\}=-\frac{1}{(k-1)!}\binom{2\nu-k}{\nu-k+1}. \tag{4.2.30}$$

By inserting the values (4.2.29) and (4.2.30) into (4.2.28), we obtain

$$(1-x)h_{\nu+1,k}(x)-h_{\nu,k}(x)=\frac{x^{\nu-k+1}}{(k-1)!}\left\{\binom{2\nu-k}{\nu-k+1}-\binom{2\nu-k+1}{\nu-k+1}x\right\}. \tag{4.2.31}$$

Now multiply both sides of (4.2.31) by $x^{k-1}(1-x)^{\nu}$, and use (4.2.11), to deduce that

$$f_{\nu+1,k}(x)-f_{\nu,k}(x)=\frac{x^{\nu}(1-x)^{\nu}}{(k-1)!}\left\{\binom{2\nu-k}{\nu-k+1}-\binom{2\nu-k+1}{\nu-k+1}x\right\}. \tag{4.2.32}$$

By combining (4.2.32), (4.1.2) and (4.2.2), we have now established the following recursive formulation.

**Theorem 4.2.2** *For $\nu = 1, 2, \cdots$, the polynomials $\{f_{\nu,k} : \ k = 1, \cdots, \nu\}$ of Theorem 4.2.1 satisfy the recursive formulation*

$$\begin{cases} f_{1,1}(x) = 1 - x; \\ f_{\nu+1,k}(x) = f_{\nu,k}(x) + \dfrac{x^\nu(1-x)^\nu}{(k-1)!}\left\{\dbinom{2\nu-k}{\nu-k+1} - \dbinom{2\nu-k+1}{\nu-k+1}x\right\}, \quad k = 1, \cdots, \nu; \\ f_{\nu+1,\nu+1}(x) = \dfrac{1}{\nu!}x^\nu(1-x)^{\nu+1}. \end{cases} \tag{4.2.33}$$

Calculating either by means of Theorem 4.2.1 or 4.2.2, we obtain Table 4.1, in which the polynomial sequences $\{f_{\nu,k} : \ k = 1, \cdots, \nu\}$ are given for $\nu = 2, 3, 4$.

*Table 4.1: The polynomials $\{f_{\nu,k} : \ k = 1, \cdots, \nu\}$ for $\nu = 2, 3, 4$*

| $\nu$ | $k$ | $\{f_{\nu,k} : \ k = 1, \cdots, \nu\}$ |
|---|---|---|
| 2 | 1 | $f_{2,1}(x) = 1 - 3x^2 + 2x^3$ |
| 2 | 2 | $f_{2,2}(x) = x - 2x^2 + x^3$ |
| 3 | 1 | $f_{3,1}(x) = -6x^5 + 15x^4 - 10x^3 + 1$ |
| 3 | 2 | $f_{3,2}(x) = -3x^5 + 8x^4 - 6x^3 + x$ |
| 3 | 3 | $f_{3,3}(x) = -\frac{1}{2}x^5 + \frac{3}{2}x^4 - \frac{3}{2}x^3 + \frac{1}{2}x^2$ |
| 4 | 1 | $f_{4,1}(x) = 20\,x^7 - 70\,x^6 + 84\,x^5 - 35\,x^4 + 1;$ |
| 4 | 2 | $f_{4,2}(x) = 10\,x^7 - 36\,x^6 + 45\,x^5 - 20\,x^4 + x;$ |
| 4 | 3 | $f_{4,3}(x) = 2\,x^7 - \frac{15}{2}\,x^6 + 10\,x^5 - 5\,x^4 + \frac{1}{2}\,x^2$ |
| 4 | 4 | $f_{4,4}(x) = \frac{1}{6}\,x^7 - \frac{2}{3}\,x^6 + x^5 - \frac{2}{3}\,x^4 + \frac{1}{6}\,x^3$ |

The graphs of the resulting vector splines $\mathbf{\Phi}_\nu^H$ for $\nu = 2, 3, 4$, are shown in Fig. 4.1-4.3.

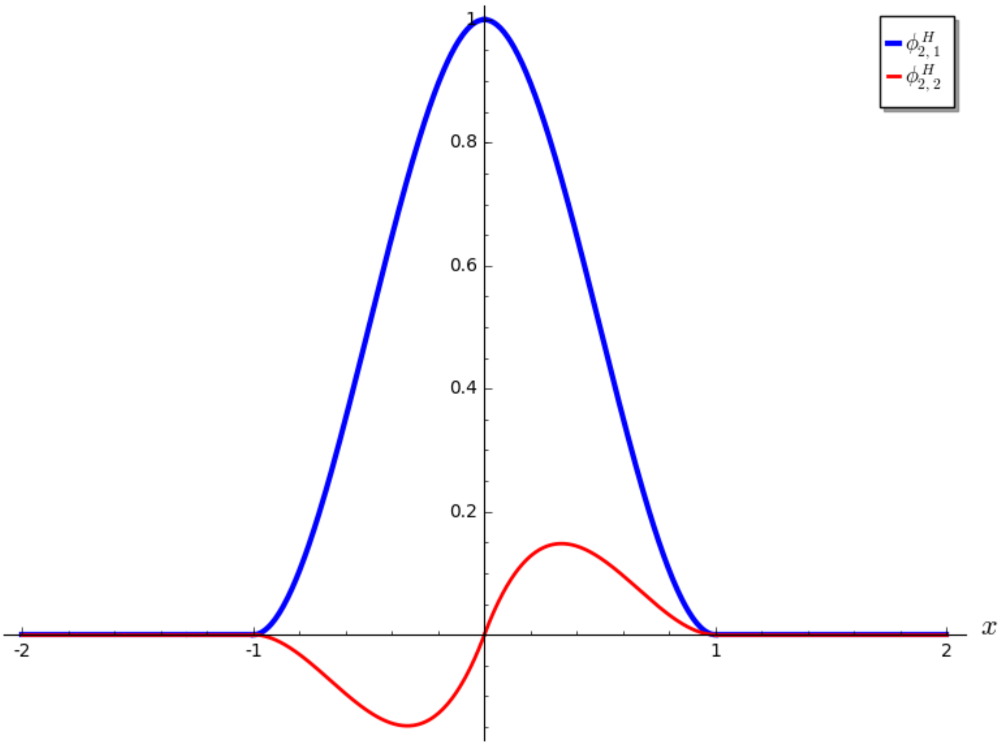



**Figure 4.1:** *The cubic Hermite vector spline* $\mathbf{\Phi}^H_2 = (\phi^H_{2,1}, \phi^H_{2,2})^T$

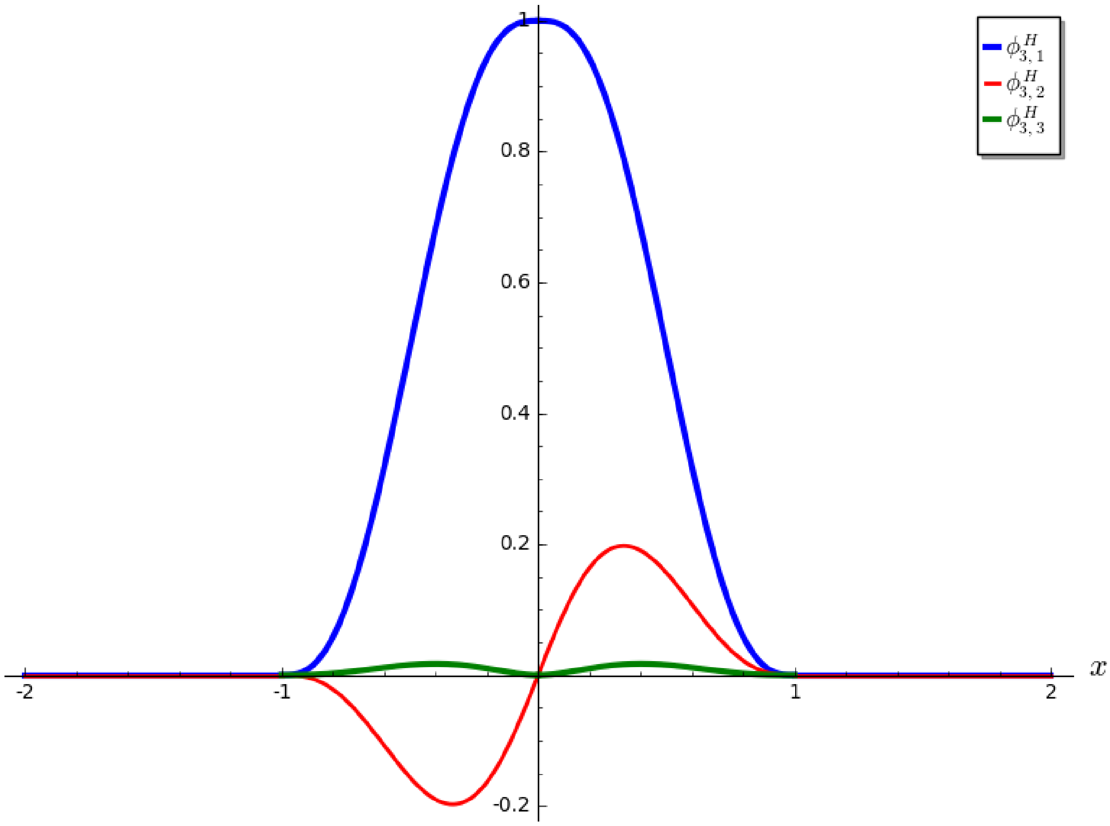
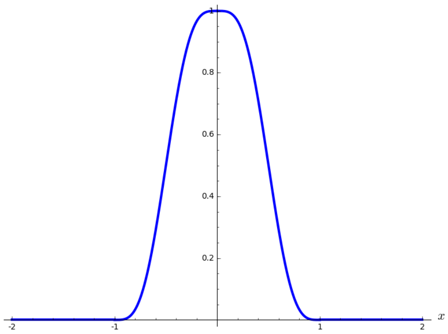
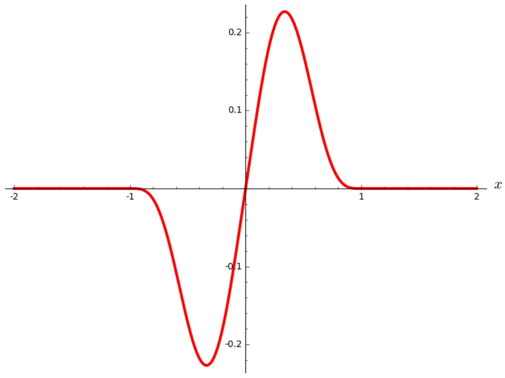


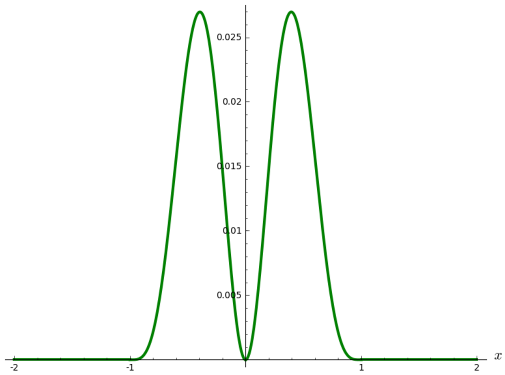

**Figure 4.2:** *The quintic Hermite vector spline* $\mathbf{\Phi}^H_3 = (\phi^H_{3,1}, \phi^H_{3,2}, \phi^H_{3,3})^T$



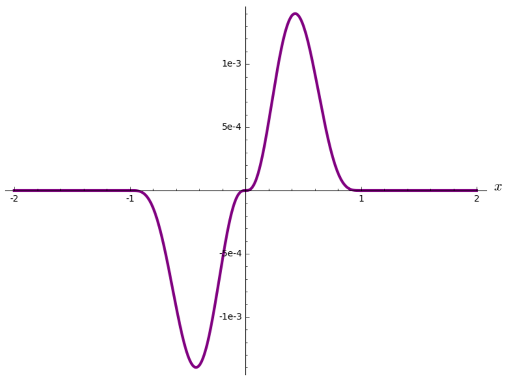

**(a)** $\phi_{4,1}^H$




**(b)** $\phi_{4,2}^H$




**(c)** $\phi_{4,3}^H$




**(d)** $\phi_{4,4}^H$

**Figure 4.3:** *The Hermite vector spline* $\mathbf{\Phi}_4^H = \left(\phi_{4,1}^H, \phi_{4,2}^H, \phi_{4,3}^H, \phi_{4,4}^H\right)^T$

## 4.3 Matrix refinement sequence computation

In order to efficiently compute the matrix refinement sequence $\{P_\nu^H(k)\}$ in Theorem 4.1.2, we first note that (4.1.10) and (4.1.6) yield

$$\left[P_\nu^H(0)\right]_{mn} = \frac{1}{2^{n-1}}\delta(n-m), \quad m,n = 1,\cdots,\nu, \tag{4.3.1}$$

that is,

$$P_\nu^H(0) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2^{\nu-1}} \end{bmatrix}. \tag{4.3.2}$$

Next, we apply (4.1.10) and (4.1.3) to deduce that

$$\left[P_\nu^H(1)\right]_{mn} = \frac{1}{2^{n-1}} f_{\nu,m}^{(n-1)}(1/2), \quad m,n = 1,\cdots,\nu, \tag{4.3.3}$$

whereas

$$\left[P_\nu^H(-1)\right]_{mn} = \frac{(-1)^{m+n}}{2^{n-1}} f_{\nu,m}^{(n-1)}(1/2), \quad m,n = 1,\cdots,\nu. \tag{4.3.4}$$

Observe from (4.3.3) and (4.3.4) that

$$\left[P_\nu^H(-1)\right]_{mn} = (-1)^{m+n}\left[P_\nu^H(1)\right]_{mn}, \quad m,n = 1,\cdots,\nu. \tag{4.3.5}$$

Hence the computation of the two matrix refinement sequences $P_\nu^H(1)$ and $P_\nu^H(-1)$ requires the evaluation of the derivatives $f_{\nu,m}^{(n-1)}(1/2), \quad m,n = 1,\cdots,\nu$, for which we now proceed to derive a recursive formulation with respect to the index $\nu$. To this end, we define the polynomial $\widetilde{f}_{\nu,k} \in \pi_{2\nu-1}$ by

$$\widetilde{f}_{\nu,k}(x) := f_{\nu,k}\left(\frac{1}{2} - x\right), \tag{4.3.6}$$

according to which

$$f_{\nu,m}^{(n-1)}(1/2) = (-1)^{n-1}\widetilde{f}_{\nu,m}^{(n-1)}(0), \quad m,n = 1,\cdots,\nu. \tag{4.3.7}$$

Since, for any $x \in \mathbb{R}$ and $k \in \{1,\cdots,\nu\}$, we have

$$\begin{aligned}\binom{2\nu-k}{\nu-k+1} - \binom{2\nu-k+1}{\nu-k+1}x &= \binom{2\nu-k}{\nu-k+1} - \left\{\binom{2\nu-k}{\nu-k+1} + \binom{2\nu-k}{\nu-k}\right\}x \\ &= \binom{2\nu-k}{\nu-k+1}(1-x) - \binom{2\nu-k}{\nu-k}x,\end{aligned}$$

we obtain, for $\nu = 1, 2, \cdots$, and from (4.3.6) and (4.2.33), the recursive formulation

$$\begin{cases} \widetilde{f}_{1,1}(x) & = \dfrac{1}{2} + x; \\ \widetilde{f}_{\nu+1,k}(x) & = \widetilde{f}_{\nu,k}(x) + \dfrac{1}{(k-1)!}\Big(\dfrac{1}{4} - x^2\Big)^{\nu}\left\{\dbinom{2\nu-k}{\nu-k+1}\Big(\dfrac{1}{2}+x\Big) - \dbinom{2\nu-k}{\nu-k}\Big(\dfrac{1}{2}-x\Big)\right\}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \cdots, \nu; \\ \widetilde{f}_{\nu+1,\nu+1}(x) & = \dfrac{1}{\nu!}\Big(\dfrac{1}{4} - x^2\Big)^{\nu}\Big(\dfrac{1}{2}+x\Big). \end{cases} \tag{4.3.8}$$

By applying the Leibniz rule for the differentiation of a product, we next deduce, for any $k \in \{1, \cdots, \nu - 1\}$ and $n \in \mathbb{N}$, that

$$\Big(\frac{d}{dx}\Big)^{n-1}\left[\Big(\frac{1}{4} - x^2\Big)^{\nu}\left\{\binom{2\nu-k}{\nu-k+1}\Big(\frac{1}{2}+x\Big) - \binom{2\nu-k}{\nu-k}\Big(\frac{1}{2}-x\Big)\right\}\right]$$
$$= \sum_{l=0}^{1}\binom{n-1}{l}\left[\Big(\frac{d}{dx}\Big)^{n-1-l}\left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\Big(\frac{1}{4}\Big)^{\nu-j}x^{2j}\right\}\right]\cdot$$
$$\Big(\frac{d}{dx}\Big)^{l}\left\{\binom{2\nu-k}{\nu-k+1}\Big(\frac{1}{2}+x\Big) - \binom{2\nu-k}{\nu-k}\Big(\frac{1}{2}-x\Big)\right\}$$
$$= \sum_{l=0}^{1}\binom{n-1}{l}\left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\Big(\frac{1}{4}\Big)^{\nu-j}\binom{2j}{n-1-l}(n-1-l)!\; x^{2j-n+1+l}\right\}\cdot$$
$$\left\{\binom{2\nu-k}{\nu-k+1}\binom{1}{l}l!\Big(\frac{1}{2}+x\Big)^{1-l} + (-1)^{l+1}\binom{2\nu-k}{\nu-k}\binom{1}{l}l!\Big(\frac{1}{2}-x\Big)^{1-l}\right\},$$

and thus

$$\left(\frac{d}{dx}\right)^{n-1}\left[\Big(\frac{1}{4} - x^2\Big)^{\nu}\left\{\binom{2\nu-k}{\nu-k+1}\Big(\frac{1}{2}+x\Big) - \binom{2\nu-k}{\nu-k}\Big(\frac{1}{2}-x\Big)\right\}\right]\Bigg|_{x=0}$$
$$= \sum_{l=0}^{1}\binom{n-1}{l}\left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\Big(\frac{1}{4}\Big)^{\nu-j}\binom{2j}{n-1-l}(n-1-l)!\delta(2j-n+1+l)\right\}\cdot$$
$$\left\{\binom{2\nu-k}{\nu-k+1}\binom{1}{l}l!\Big(\frac{1}{2}\Big)^{1-l} + (-1)^{l+1}\binom{2\nu-k}{\nu-k}\binom{1}{l}l!\Big(\frac{1}{2}\Big)^{1-l}\right\}$$

$$\left(\frac{d}{dx}\right)^{n-1}\left[\Big(\frac{1}{4} - x^2\Big)^{\nu}\left\{\binom{2\nu-k}{\nu-k+1}\Big(\frac{1}{2}+x\Big) - \binom{2\nu-k}{\nu-k}\Big(\frac{1}{2}-x\Big)\right\}\right]\Bigg|_{x=0}$$
$$= \left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\Big(\frac{1}{4}\Big)^{\nu-j}\binom{2j}{n-1}(n-1)!\;\delta(2j-n+1)\right\}\left\{\binom{2\nu-k}{\nu-k+1}\Big(\frac{1}{2}\Big) - \binom{2\nu-k}{\nu-k}\Big(\frac{1}{2}\Big)\right\}$$
$$+ (n-1)\left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\Big(\frac{1}{4}\Big)^{\nu-j}\binom{2j}{n-2}(n-2)!\;\delta(2j-n+2)\right\}\left\{\binom{2\nu-k}{\nu-k+1} + \binom{2\nu-k}{\nu-k}\right\},$$

from which it then follows that

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^{2n-1}\left[\left(\frac{1}{4}-x^2\right)^{\nu}\left\{\binom{2\nu-k}{\nu-k+1}\left(\frac{1}{2}+x\right)-\binom{2\nu-k}{\nu-k}\left(\frac{1}{2}-x\right)\right\}\right]\Bigg|_{x=0} \\
&\quad = \frac{1}{2}\left\{\binom{2\nu-k}{\nu-k+1}-\binom{2\nu-k}{\nu-k}\right\}\left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\left(\frac{1}{4}\right)^{\nu-j}\binom{2j}{n-1}(n-1)!\ \delta(2j-n+1)\right\} \\
&\qquad + (n-1)\binom{2\nu-k+1}{\nu-k+1}\left\{\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\left(\frac{1}{4}\right)^{\nu-j}\binom{2j}{n-2}(n-2)!\ \delta(2j-n+2)\right\} \\
&\quad = (2n-1)\binom{2\nu-k+1}{\nu-k+1}(-1)^{n-1}\binom{\nu}{n-1}\left(\frac{1}{4}\right)^{\nu-n+1}(2n-2)! \\
&\quad = (-1)^{n-1}\frac{(2n-1)!}{2^{2\nu-2n+2}}\binom{2\nu-k+1}{\nu-k+1}\binom{\nu}{n-1}, \qquad (4.3.9)
\end{aligned}$$

whereas

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^{2n}\left[\left(\frac{1}{4}-x^2\right)^{\nu}\left\{\binom{2\nu-k}{\nu-k+1}\left(\frac{1}{2}+x\right)-\binom{2\nu-k}{\nu-k}\left(\frac{1}{2}-x\right)\right\}\right]\Bigg|_{x=0} \\
&\qquad = \frac{1}{2}\left\{\binom{2\nu-k}{\nu-k+1}-\binom{2\nu-k}{\nu-k}\right\}(-1)^n\binom{\nu}{n}\left(\frac{1}{4}\right)^{\nu-n}(2n)! \\
&\qquad = (-1)^n\frac{(2n)!}{2^{2\nu-2n+1}}\left\{\binom{2\nu-k}{\nu-k+1}-\binom{2\nu-k}{\nu-k}\right\}\binom{\nu}{n}. \qquad (4.3.10)
\end{aligned}$$

A similar calculation yields

$$\left(\frac{d}{dx}\right)^{2n-1}\left\{\left(\frac{1}{4}-x^2\right)^{\nu}\left(\frac{1}{2}+x\right)\right\}\Big|_{x=0} = (-1)^{n-1}\frac{(2n-1)!}{2^{2\nu-2n+2}}\binom{\nu}{n-1}; \qquad (4.3.11)$$

$$\left(\frac{d}{dx}\right)^{2n}\left\{\left(\frac{1}{4}-x^2\right)^{\nu}\left(\frac{1}{2}+x\right)\right\}\Big|_{x=0} = (-1)^{n}\frac{(2n)!}{2^{2\nu-2n+1}}\binom{\nu}{n}. \qquad (4.3.12)$$

Hence we may now apply (4.1.2) and (4.3.7)-(4.3.12) to state the following algorithm for the recursive computation of the derivatives $\left\{f_{\nu,m}^{(n-1)}(1/2):\ m,n=1,\cdots,\nu\right\}$.

*Algorithm 4.1:* *Recursive computation of the matrix refinement sequence* $\left\{P_\nu^H(k),\ k\in\{-1,1\}\right\}$ *for* $\nu=1,2,\cdots$.

Let $\nu\in\mathbb{N}$ be given.

1. Set

$$\widetilde{f}_{1,1}(0)=\frac{1}{2};\quad \widetilde{f}'_{1,1}(0)=1.$$

2. For $\nu=1,2,\cdots$, set

$$\widetilde{f}^{(n-1)}_{\nu+1,m}(0)=\widetilde{f}^{(n-1)}_{\nu,m}(0)+\frac{1}{(m-1)!}\begin{cases}\dfrac{(-1)^{l-1}(n-1)!}{2^{2\nu-2l+2}}\dbinom{2\nu-m+1}{\nu-m+1}\dbinom{\nu}{l-1}, & \text{if } n=2l;\\[2ex] \dfrac{(-1)^{l}(n-1)!}{2^{2\nu-2l+1}}\left[\dbinom{2\nu-m}{\nu-m+1}-\dbinom{2\nu-m}{\nu-m}\right]\dbinom{\nu}{l}, & \\ & \text{if } n=2l+1,\end{cases}$$

$$\text{for } n=1,\cdots,\nu+1,\quad m=1,\cdots,\nu;$$

$$\widetilde{f}^{(n-1)}_{\nu+1,\nu+1}(0)=\frac{1}{\nu!}\begin{cases}\dfrac{(-1)^{l-1}(n-1)!}{2^{2\nu-2l+2}}\dbinom{\nu}{l-1}, & \text{if } n=2l;\\[2ex] \dfrac{(-1)^{l}(n-1)!}{2^{2\nu-2l+1}}\dbinom{\nu}{l}, & \text{if } n=2l+1,\end{cases}\qquad \text{for } n=1,\cdots,\nu+1.$$

3. For $m,n=1,\cdots,\nu$, set

$$\left[P_\nu^H(1)\right]_{mn}=\frac{(-1)^{n-1}}{2^{n-1}}\widetilde{f}^{(n-1)}_{\nu,m}(0);$$
$$\left[P_\nu^H(-1)\right]_{mn}=(-1)^{m+n}\left[P_\nu^H(1)\right]_{mn}.$$

**Remark 4.3.1** Observe from (4.2.3) and (4.2.7) that, for any integer $\nu\geq 2$,

$$f_{\nu,1}(x)+f_{\nu,1}(1-x)=1,\quad x\in\mathbb{R},$$

and thus

$$\{1+(-1)^l\}f^{(l)}_{\nu,1}(1/2)=0,\quad l=1,\cdots,\nu-1,$$

from which we deduce that

$$f^{(2n)}_{\nu,1}(1/2)=0,\quad n=1,\cdots,\left\lfloor\frac{1}{2}(\nu-1)\right\rfloor\quad (\text{if } \nu\geq 3). \tag{4.3.13}$$

It then follows from (4.3.13), (4.3.3) and (4.3.5) that

$$\left[P_\nu^H(-1)\right]_{1,2n+1} = \left[P_\nu^H(1)\right]_{1,2n+1} = 0, \quad n = 1, \cdots, \left\lfloor \frac{1}{2}(\nu-1) \right\rfloor \quad (\text{if } \nu \geq 3). \qquad (4.3.14)$$

Hence Algorithm 4.1 will yield alternate zeros according to (4.3.14) in the first row of both $P_\nu^H(1)$ and $P_\nu^H(-1)$. ■

Calculating by means of Algorithm 4.1, together with (4.3.2)-(4.3.4), we now obtain Table 4.2, in which the matrix refinement sequence $\{P_\nu^H(k): \ k = -1, 0, 1\}$ is listed for $\nu = 2, 3, 4$.

*Table 4.2: Matrix refinement sequence $\{P_\nu^H(k): \ k = -1, 0, 1\}$ for $\nu = 2, 3, 4$*

| $\nu$ | $\{P_\nu^H(k): \ k = -1, 0, 1\}$ |
|---|---|
| 2 | $P_2^H(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad P_2^H(1) = \begin{bmatrix} 1/2 & -3/4 \\ 1/8 & -1/8 \end{bmatrix}, \quad P_2^H(-1) = \begin{bmatrix} 1/2 & 3/4 \\ -1/8 & 1/8 \end{bmatrix}.$ |
| 3 | $P_3^H(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \quad P_3^H(1) = \begin{bmatrix} 1/2 & -15/16 & 0 \\ 5/32 & -7/32 & -3/8 \\ 1/64 & -1/64 & -1/16 \end{bmatrix},$ $P_3^H(-1) = \begin{bmatrix} 1/2 & 15/16 & 0 \\ -5/32 & -7/32 & 3/8 \\ 1/64 & 1/64 & -1/16 \end{bmatrix}.$ |
| 4 | $P_4^H(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix}, \quad P_4^H(1) = \begin{bmatrix} \frac{1}{2} & -\frac{35}{16} & 0 & \frac{105}{2} \\ \frac{11}{64} & -\frac{19}{32} & -\frac{15}{8} & \frac{105}{4} \\ \frac{3}{128} & -\frac{1}{16} & -\frac{7}{16} & \frac{15}{4} \\ \frac{1}{768} & -\frac{1}{384} & -\frac{1}{32} & \frac{3}{16} \end{bmatrix},$ $P_4^H(-1) = \begin{bmatrix} \frac{1}{2} & \frac{35}{16} & 0 & -\frac{105}{2} \\ -\frac{11}{64} & -\frac{19}{32} & \frac{15}{8} & \frac{105}{4} \\ \frac{3}{128} & \frac{1}{16} & -\frac{7}{16} & -\frac{15}{4} \\ -\frac{1}{768} & -\frac{1}{384} & \frac{1}{32} & \frac{3}{16} \end{bmatrix}.$ |

## 4.4 Integer-shift linear independence and stability

We proceed to establish the following linear independence and stability result.

**Theorem 4.4.1** *For any $\nu \in \mathbb{N}$, the Hermite interpolation vector spline $\mathbf{\Phi}_\nu^H = (\phi_{\nu,1}^H, \cdots, \phi_{\nu,\nu}^H)^T$ of Theorem 4.1.2 possesses matrix linearly independent integer shifts, as well as $l^2$-stable integer shifts, on $\mathbb{R}$.*

**Proof.** Let $\{c_1(k)\}, \cdots, \{c_\nu(k)\}$ denote any sequences in $l(\mathbb{Z})$ such that

$$\sum_{j=1}^{\nu} \sum_k c_j(k)\phi_{\nu,j}^H(x-k) = 0, \quad x \in \mathbb{R}. \tag{4.4.1}$$

By recalling also (4.1.4), it follows from (4.4.1) that

$$\sum_{j=1}^{\nu} \sum_k c_j(k)\big(\phi_{\nu,j}^H\big)^{(l)}(x-k) = 0, \quad x \in \mathbb{R}, \quad l = 0, \cdots, \nu-1, \tag{4.4.2}$$

and thus

$$\sum_{j=1}^{\nu} \sum_k c_j(k)\big(\phi_{\nu,j}^H\big)^{(l)}(\mu-k) = 0, \quad \mu \in \mathbb{Z}, \quad l = 0, \cdots, \nu-1. \tag{4.4.3}$$

Hence we may substitute (4.1.6) into (4.4.3) to obtain

$$\sum_{j=1}^{\nu} \sum_k c_j(k)\delta(l+1-j)\delta(\mu-k) = 0, \quad \mu \in \mathbb{Z}, \quad l = 0, \cdots, \nu-1,$$

which yields

$$c_{l+1}(\mu) = 0, \quad \mu \in \mathbb{Z}, \quad l = 0, \cdots, \nu-1,$$

or equivalently,

$$c_j(k) = 0, \quad k \in \mathbb{Z}, \quad j = 1, \cdots, \nu. \tag{4.4.4}$$

Hence (4.4.1) implies (4.4.4), which shows that $\mathbf{\Phi}_\nu^H$ possesses matrix linearly independent integer shifts on $\mathbb{R}$. The $l^2$-stability statement of the theorem is now an immediate consequence of Theorem 1.5.2. ■

In this chapter, we investigated the class of Hermite refinable vector splines on the interval $[-1, 1]$, and in particular derived, as our contribution to this topic, explicit expressions for these splines and matrix refinement sequences. In the next Chapter 5, our multi-wavelet construction method will be presented as an extension of the (scalar) wavelet construction method in [3]. Also, explicit spline multi-wavelets will then be produced from the refinable vector splines given in, respectively, Chapters 4 and 3.

# Chapter 5

# MULTI-WAVELET CONSTRUCTION

In this chapter, by extending the method applied in [3] in the scalar case, we construct multi-wavelets with shortest matrix filters from the building blocks of refinable vector functions with matrix linearly independent integer shifts. Our characterisation of such multi-wavelets are in terms of identity systems satisfied by matrix Laurent polynomials. The resulting spline multi-wavelets based on the refinable vector splines of Chapters 3 and 4 are then presented in, respectively, Sections 5.3 and 5.2.

## 5.1 Refinement space decomposition from matrix Laurent polynomial identities

Let $\mathbf{\Phi}$ denote any refinable vector function of length $\nu \in \mathbb{N}$. We now define the vector refinement space sequence

$$\mathbf{S}_{\mathbf{\Phi}}^{r} := \Big\{ \sum_{j} M(j)\mathbf{\Phi}(2^{r} \cdot -j) : \ \{M(j)\} \in l^{\nu \times \nu}(\mathbb{Z}) \Big\}, \ r \in \mathbb{Z}. \tag{5.1.1}$$

We shall write $\mathbf{S}_{\mathbf{\Phi}}$ for $\mathbf{S}_{\mathbf{\Phi}}^{0}$.

The vector refinability of $\mathbf{\Phi}$ then yields the following nesting property.

**Theorem 5.1.1** *For any refinable vector function $\mathbf{\Phi}$ of length $\nu \in \mathbb{N}$, the vector refinement space sequence $\Big\{\mathbf{S}_{\mathbf{\Phi}}^{r} : \ r \in \mathbb{Z}\Big\}$, as defined by* (5.1.1)*, satisfies the nesting property*

$$\mathbf{S}_{\mathbf{\Phi}}^{r} \subset \mathbf{S}_{\mathbf{\Phi}}^{r+1}, \ r \in \mathbb{Z}, \tag{5.1.2}$$

*with, more precisely, for any matrix sequence $\{M(j)\} \in l^{\nu \times \nu}(\mathbb{Z})$,*

$$\sum_{j} M(j)\mathbf{\Phi}(2^{r}x - j) = \sum_{j} \widetilde{M}(j)\mathbf{\Phi}(2^{r+1}x - j), \quad x \in \mathbb{R}, \quad r \in \mathbb{Z}, \tag{5.1.3}$$

*where the matrix sequence* $\{\widetilde{M}(j)\} \in l^{\nu\times\nu}(\mathbb{Z})$ *is defined by*

$$\widetilde{M}(j) := \sum_{l} M(l)P(j-2l), \quad j \in \mathbb{Z}, \tag{5.1.4}$$

*and where* $\{P(k)\}$ *is the matrix refinement sequence of* $\mathbf{\Phi}$.

**Proof.** For any matrix sequence $\{M(j)\} \in l^{\nu\times\nu}(\mathbb{Z})$, it follows from the vector refinement equation (1.3.1) that, for any $x \in \mathbb{R}$ and $r \in \mathbb{Z}$,

$$\begin{aligned}\sum_{j} M(j)\mathbf{\Phi}(2^r x - j) &= \sum_{j} M(j) \sum_{l} P(l)\mathbf{\Phi}(2^{r+1}x - 2j - l)\\ &= \sum_{j} M(j) \sum_{l} P(l-2j)\mathbf{\Phi}(2^{r+1}x - l)\\ &= \sum_{l} \Big\{ \sum_{j} M(j)P(l-2j) \Big\} \mathbf{\Phi}(2^{r+1}x - l),\end{aligned}$$

which proves (5.1.3)-(5.1.4), and therefore also the desired nesting property (5.1.2).

■

For a given linear space $W$, if $U$ and $V$ are two subspaces of $W$ with the property that for each $w \in W$ there exist elements $u \in U$ and $v \in V$ such that $w = u + v$, and with the pair $\{u, v\}$ uniquely determined by $w$, we say that $W$ is the direct sum of $U$ and $V$, and we write

$$W = U \oplus V. \tag{5.1.5}$$

We shall rely on the fact that the mere existence of the pair $\{u, v\}$, together with the property

$$U \cap V = \{0\}, \tag{5.1.6}$$

with 0 denoting the zero element of the space $W$, implies that $\{u, v\}$ is uniquely determined by $w$. Indeed, if $w = u+v = \tilde{u}+\tilde{v}$, with $\tilde{u} \in U$ and $\tilde{v} \in V$, then $u-\tilde{u} = v-\tilde{v} \in U\cap V = \{0\}$, according to which we must have $\tilde{u} = u$ and $\tilde{v} = v$.

Now observe from (5.1.1) that

$$\mathbf{\Psi} \in \mathbf{S}^1_{\mathbf{\Phi}} \cap \{f :\ \mathbb{R} \to \mathbb{R}, \quad \text{with } f \text{ compactly supported}\}. \tag{5.1.7}$$

if, for some matrix sequence $Q = \{Q(j)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$, we have

$$\mathbf{\Psi}(x) = \mathbf{\Psi}_{\mathbf{\Phi},Q}(x) := \sum_{j} Q(j)\mathbf{\Phi}(2x - j). \tag{5.1.8}$$

With the space sequence definition

$$\mathbf{W}^r_{\mathbf{\Phi},Q} := \Big\{ \sum_j D(j)\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r \cdot -j) : \ \{D(j)\} \in l^{\nu\times\nu}(\mathbb{Z}) \Big\}, \quad r \in \mathbb{Z}, \tag{5.1.9}$$

we observe that (5.1.8) and (5.1.1) imply

$$\mathbf{W}^r_{\mathbf{\Phi},Q} \subset \mathbf{S}^{r+1}_{\mathbf{\Phi}}, \quad r \in \mathbb{Z}. \tag{5.1.10}$$

More precisely, we have, analogously to Theorem 5.1.1, the following result.

**Theorem 5.1.2** *For any refinable vector function* $\mathbf{\Phi}$ *of length* $\nu \in \mathbb{N}$, *let* $\mathbf{\Psi} = \mathbf{\Psi}_{\mathbf{\Phi},Q}$ *denote a compactly supported vector function in* $S^1_{\mathbf{\Phi}}$, *as defined by* (5.1.8) *for some sequence* $\{Q(j)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$. *Then, for any matrix sequence* $\{M(j)\} \in l^{\nu\times\nu}(\mathbb{Z})$,

$$\sum_j M(j)\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r x - j) = \sum_j \widetilde{M}(j)\mathbf{\Phi}(2^{r+1}x - j), \quad x \in \mathbb{R}, \quad r \in \mathbb{Z}, \tag{5.1.11}$$

*where the matrix sequence* $\{\widetilde{M}(j)\} \in l^{\nu\times\nu}(\mathbb{Z})$ *is defined by*

$$\widetilde{M}(j) := \sum_l M(l)Q(j-2l), \quad j \in \mathbb{Z}. \tag{5.1.12}$$

**Proof.** For any matrix sequence $\{M(j)\} \in l^{\nu\times\nu}(\mathbb{Z})$, it follows from the definition (5.1.8) of the vector function $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ that, for any $x \in \mathbb{R}$ and $r \in \mathbb{Z}$,

$$\begin{aligned}
\sum_j M(j)\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r x - j) &= \sum_j M(j) \sum_l Q(l)\mathbf{\Phi}(2^{r+1}x - 2j - l) \\
&= \sum_j M(j) \sum_l Q(l-2j)\mathbf{\Phi}(2^{r+1}x - l) \\
&= \sum_l \Big\{ \sum_j M(j)Q(l-2j) \Big\} \mathbf{\Phi}(2^{r+1}x - l),
\end{aligned}$$

which proves (5.1.11), (5.1.12). ■

Our objective is to find a vector function $\mathbf{\Psi} = \mathbf{\Psi}_{\mathbf{\Phi},Q}$ as in (5.1.7)-(5.1.8) such that the space decomposition

$$\mathbf{S}^{r+1}_{\mathbf{\Phi}} = \mathbf{S}^r_{\mathbf{\Phi}} \oplus \mathbf{W}^r_{\mathbf{\Phi},Q} \tag{5.1.13}$$

is achieved, and in which case we shall call $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ a multi-wavelet. In view of (5.1.1), (5.1.9) and (5.1.10), our first step is to seek three matrix sequences

$$\Big\{ \{Q(k)\}, \ \{A(k)\}, \ \{B(k)\} \Big\} \subset l_0^{\nu\times\nu}(\mathbb{Z}) \tag{5.1.14}$$

such that, with $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ as in (5.1.8), the property

$$\mathbf{\Phi}(2x-j) = \sum_l A(2l-j)\mathbf{\Phi}(x-l) + \sum_l B(2l-j)\mathbf{\Psi}_{\mathbf{\Phi},Q}(x-l), \quad x \in \mathbb{R}, \quad j \in \mathbb{Z} \tag{5.1.15}$$

is satisfied. Our result in Theorem 5.1.3 below, which extends the scalar result [3, Theorem 9.1.2] to the vector setting, is formulated in terms of the matrix Laurent polynomials

$$\mathcal{P}(z) := \frac{1}{2}\sum_k P(k)z^k; \tag{5.1.16}$$

$$\mathcal{Q}(z) := \frac{1}{2}\sum_k Q(k)z^k; \tag{5.1.17}$$

$$\mathcal{A}(z) := \sum_k A(k)z^k; \tag{5.1.18}$$

$$\mathcal{B}(z) := \sum_k B(k)z^k. \tag{5.1.19}$$

**Theorem 5.1.3** *Let $\mathbf{\Phi}$ be a refinable vector function of length $\nu \in \mathbb{N}$, with matrix refinement sequence $\{P(k)\}$ and corresponding matrix Laurent polynomial symbol $\mathcal{P}$ defined by* (5.1.16)*. Suppose, moreover, that $\mathbf{\Phi}$ possesses matrix linearly independent integer shifts on $\mathbb{R}$. Then three matrix sequences $\{Q(k)\}$, $\{A(k)\}$, $\{B(k)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$ satisfy the identity* (5.1.15)*, with the function $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ defined as in* (5.1.8)*, if and only if the corresponding matrix Laurent polynomials $\mathcal{Q}$, $\mathcal{A}$ and $\mathcal{B}$, as given in* (5.1.17)-(5.1.19)*, satisfy any one of the two equivalent systems*

$$\left.\begin{aligned} \mathcal{A}(z)\mathcal{P}(z) + \mathcal{B}(z)\mathcal{Q}(z) &= I; \\ \mathcal{A}(-z)\mathcal{P}(z) + \mathcal{B}(-z)\mathcal{Q}(z) &= O; \end{aligned}\right\} \tag{5.1.20}$$

$$\left.\begin{aligned} \mathcal{P}(z)\mathcal{A}(z) + \mathcal{P}(-z)\mathcal{A}(-z) &= I; \\ \mathcal{Q}(z)\mathcal{A}(z) + \mathcal{Q}(-z)\mathcal{A}(-z) &= O; \\ \mathcal{P}(z)\mathcal{B}(z) + \mathcal{P}(-z)\mathcal{B}(-z) &= O; \\ \mathcal{Q}(z)\mathcal{B}(z) + \mathcal{Q}(-z)\mathcal{B}(-z) &= I. \end{aligned}\right\} \tag{5.1.21}$$

**Proof.** For any matrix sequences $\Big\{\{A(k)\},\ \{B(k)\},\ \{Q(k)\}\Big\} \subset l_0^{\nu\times\nu}(\mathbb{Z})$, it follows from (1.3.1) and (5.1.8) that, for any $x \in \mathbb{R}$,

$$\begin{aligned}
&\sum_l A(2l-j)\mathbf{\Phi}(x-l) + \sum_l B(2l-j)\mathbf{\Psi}_{\mathbf{\Phi},Q}(x-l)\\
&\qquad = \sum_l A(2l-j)\Big[\sum_m P(m)\mathbf{\Phi}(2x-2l-m)\Big] + \sum_l B(2l-j)\Big[\sum_m Q(m)\mathbf{\Phi}(2x-2l-m)\Big]\\
&\qquad = \sum_l A(2l-j)\Big[\sum_m P(m-2l)\mathbf{\Phi}(2x-m)\Big] + \sum_l B(2l-j)\Big[\sum_m Q(m-2l)\mathbf{\Phi}(2x-m)\Big]\\
&\qquad = \sum_m \Big[\sum_l A(2l-j)P(m-2l)\Big]\mathbf{\Phi}(2x-m) + \sum_m \Big[\sum_l B(2l-j)Q(m-2l)\Big]\mathbf{\Phi}(2x-m),
\end{aligned}$$

according to which (5.1.15) has the equivalent formulation

$$\sum_m \Big[\sum_l A(2l-j)P(m-2l) + \sum_l B(2l-j)Q(m-2l) - \delta(j-m)I\Big]\mathbf{\Phi}(2x-m) = \mathbf{0}, \quad j \in \mathbb{Z}. \tag{5.1.22}$$

Since, the vector function $\mathbf{\Phi}$ possesses matrix linearly independent integer shifts on $\mathbb{R}$, it follows that (5.1.22) holds if and only if

$$\sum_l A(2l-j)P(m-2l) + \sum_l B(2l-j)Q(m-2l) = \delta(j-m)I, \quad j,m \in \mathbb{Z}. \tag{5.1.23}$$

For any $z \in \mathbb{C}\setminus\{0\}$ and $j \in \mathbb{Z}$, it follows from (5.1.16)-(5.1.19) that (5.1.23) is equivalent to

$$\begin{aligned}
z^j\, I &= \sum_m \delta(j-m)Iz^m\\
&= \sum_m \Big[\sum_l A(2l-j)P(m-2l) + \sum_l B(2l-j)Q(m-2l)\Big]z^m\\
&= \sum_m\sum_l A(2l-j)P(m-2l)z^{m-2l}z^{2l} + \sum_m\sum_l B(2l-j)Q(m-2l)z^{m-2l}z^{2l}\\
&= \sum_l A(2l-j)\Big[\sum_m P(m-2l)z^{m-2l}\Big]\, z^{2l} + \sum_l B(2l-j)\Big[\sum_m Q(m-2l)z^{m-2l}\Big]z^{2l}\\
&= \sum_l A(2l-j)\Big[\sum_m P(m)z^m\Big]z^{2l} + \sum_l B(2l-j)\Big[\sum_m Q(m)z^m\Big]z^{2l}\\
&= 2z^j\,\Big[\Big(\sum_l A(2l-j)z^{2l-j}\Big)\mathcal{P}(z) + \Big(\sum_l B(2l-j)z^{2l-j}\Big)\mathcal{Q}(z)\Big],
\end{aligned}$$

or equivalently,

$$\Big(\sum_l A(2l-j)z^{2l-j}\Big)\mathcal{P}(z) + \Big(\sum_l B(2l-j)z^{2l-j}\Big)\mathcal{Q}(z) = \frac{1}{2}\, I, \quad j \in \mathbb{Z}. \tag{5.1.24}$$

If $j = 2m$ for some $m \in \mathbb{Z}$, then (5.1.24) yields

$$\Big(\sum_l A(2l-2m)z^{2l-2m}\Big)\mathcal{P}(z) + \Big(\sum_l B(2l-2m)z^{2l-2m}\Big)\mathcal{Q}(z) = \frac{1}{2}\,I, \tag{5.1.25}$$

whereas if $j = 2m+1$ for some $m \in \mathbb{Z}$, then (5.1.24) yields

$$\Big(\sum_l A(2l-2m-1)z^{2l-2m-1}\Big)\mathcal{P}(z) + \Big(\sum_l B(2l-2m-1)z^{2l-2m-1}\Big)\mathcal{Q}(z) = \frac{1}{2}\,I. \tag{5.1.26}$$

It follows from (5.1.25) and (5.1.26) that the condition (5.1.24) is equivalent to the system

$$\left.\begin{aligned} &\Big(\sum_l A(2l)z^{2l}\Big)\mathcal{P}(z) + \Big(\sum_l B(2l)z^{2l}\Big)\mathcal{Q}(z) = \frac{1}{2}\,I;\\ &\Big(\sum_l A(2l+1)z^{2l+1}\Big)\mathcal{P}(z) + \Big(\sum_l B(2l+1)z^{2l+1}\Big)\mathcal{Q}(z) = \frac{1}{2}\,I; \end{aligned}\right\} \tag{5.1.27}$$

or equivalently, from (5.1.18) and (5.1.19),

$$\left.\begin{aligned} &[\mathcal{A}(z)+\mathcal{A}(-z)]\mathcal{P}(z) + [\mathcal{B}(z)+\mathcal{B}(-z)]\mathcal{Q}(z) = I;\\ &[\mathcal{A}(z)-\mathcal{A}(-z)]\mathcal{P}(z) + [\mathcal{B}(z)-\mathcal{B}(-z)]\mathcal{Q}(z) = I. \end{aligned}\right\} \tag{5.1.28}$$

By successively adding and subtracting the two identities in the system (5.1.28), we deduce that (5.1.28) is equivalent to the system (5.1.20).

Next, we replace $z$ by $-z$ in (5.1.20) to deduce that (5.1.20) is equivalent to the system

$$\begin{cases} \mathcal{A}(z)\mathcal{P}(z) + \mathcal{B}(z)\mathcal{Q}(z) = I;\\ \mathcal{A}(-z)\mathcal{P}(z) + \mathcal{B}(-z)\mathcal{Q}(z) = O;\\ \mathcal{A}(-z)\mathcal{P}(-z) + \mathcal{B}(-z)\mathcal{Q}(-z) = I;\\ \mathcal{A}(z)\mathcal{P}(-z) + \mathcal{B}(z)\mathcal{Q}(-z) = O, \end{cases}$$

which can be reformulated as the block matrix identity

$$\begin{bmatrix} \mathcal{A}(z) & \mathcal{B}(z)\\ \mathcal{A}(-z) & \mathcal{B}(-z) \end{bmatrix} \begin{bmatrix} \mathcal{P}(z) & \mathcal{P}(-z)\\ \mathcal{Q}(z) & \mathcal{Q}(-z) \end{bmatrix} = \begin{bmatrix} I & O\\ O & I \end{bmatrix}. \tag{5.1.29}$$

For any two (square) matrices $M,\ N \in \mathcal{M}_\nu$, a standard result in linear algebra states that $MN = I$ if and only if $NM = I$. Since the two block matrices in the left-hand side of (5.1.29) may, for any fixed $z \in \mathbb{C}\setminus\{0\}$, be interpreted as two square matrices in $\mathcal{M}_{2\nu}$, and since such block matrix multiplication is consistent with matrix multiplication in $\mathcal{M}_{2\nu}$, we deduce that the identity (5.1.29) holds if and only if

$$\begin{bmatrix} \mathcal{P}(z) & \mathcal{P}(-z) \\ \mathcal{Q}(z) & \mathcal{Q}(-z) \end{bmatrix} \begin{bmatrix} \mathcal{A}(z) & \mathcal{B}(z) \\ \mathcal{A}(-z) & \mathcal{B}(-z) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \tag{5.1.30}$$

which is precisely the system (5.1.21), and thereby completing our proof. ■

**Remark 5.1.1** Note that the case $\nu = 1$ of Theorem 5.1.3 corresponds precisely with the scalar wavelet construction in [3, Section 9.2].

Our fundamental multi-wavelet decomposition result, which extends the scalar result [3, Theorem 9.1.3] to the vector setting, is now as follows.

**Theorem 5.1.4** *Let $\mathbf{\Phi}$ be a refinable vector function of length $\nu \in \mathbb{N}$, with matrix refinement sequence $\{P(k)\}$ and corresponding matrix Laurent polynomial $\mathcal{P}$ as in* (5.1.16). *Suppose, moreover, that $\mathbf{\Phi}$ possesses matrix linearly independent integer shifts on $\mathbb{R}$. Also, let $\{Q(k)\}$, $\{A(k)\}$, and $\{B(k)\}$ denote three matrix sequences in $l_0^{\nu\times\nu}(\mathbb{Z})$ for which the corresponding matrix Laurent polynomials $\mathcal{Q}$, $\mathcal{A}$ and $\mathcal{B}$ in* (5.1.17)-(5.1.19), *satisfy either one of the two equivalent systems* (5.1.20) *and* (5.1.21) *in Theorem* 5.1.3. *Then the two space sequences $\{\mathbf{S}_{\mathbf{\Phi}}^r : r \in \mathbb{Z}\}$ and $\{\mathbf{W}_{\mathbf{\Phi},Q}^r : r \in \mathbb{Z}\}$, as defined in, respectively,* (5.1.1) *and* (5.1.9), *with the function $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ defined as in* (5.1.8), *satisfy the space decomposition result*

$$\mathbf{S}_{\mathbf{\Phi}}^{r+1} = \mathbf{S}_{\mathbf{\Phi}}^r \oplus \mathbf{W}_{\mathbf{\Phi},Q}^r, \quad r \in \mathbb{Z}, \tag{5.1.31}$$

*with, more precisely, for any matrix sequence $\{M(k)\} \in l^{\nu\times\nu}(\mathbb{Z})$, corresponding decomposition relation given by*

$$\sum_k M(k)\mathbf{\Phi}(2^{r+1}x - k) = \sum_k M^*(k)\mathbf{\Phi}(2^r x - k) + \sum_k M^{**}(k)\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r x - k), \quad r \in \mathbb{Z}, \tag{5.1.32}$$

*where*

$$M^*(k) := \sum_j M(j)A(2k - j), \quad k \in \mathbb{Z}; \tag{5.1.33}$$

$$M^{**}(k) := \sum_j M(j)B(2k - j), \quad k \in \mathbb{Z}. \tag{5.1.34}$$

*Moreover,*

$$\mathbf{S}_{\mathbf{\Phi}}^r \cap \mathbf{W}_{\mathbf{\Phi},Q}^r = \{0\}, \quad r \in \mathbb{Z}, \tag{5.1.35}$$

*and $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ is a multi-wavelet.*

**Proof.** Let $\{M(k)\} \in l^{\nu\times\nu}(\mathbb{Z})$. By applying Theorem 5.1.3, we deduce from (5.1.15) that, for any $x \in \mathbb{R}$ and $r \in \mathbb{Z}$,

$$\begin{aligned}
&\sum_k M(k)\mathbf{\Phi}(2^{r+1}x-k)\\
&\qquad = \sum_k M(k)\Big\{\sum_j A(2j-k)\mathbf{\Phi}(2^r x-j) + \sum_j B(2j-k)\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r x-j)\Big\}\\
&\qquad = \sum_j \Big[\sum_k M(k)A(2j-k)\Big]\mathbf{\Phi}(2^r x-j) + \sum_j \Big[\sum_k M(k)B(2j-k)\Big]\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r x-j),
\end{aligned}$$

which yields (5.1.32)-(5.1.34).

It remains to prove the property (5.1.35), which, according to the argument following (5.1.5), will then yield the desired space decomposition result (5.1.31), and according to which $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ is then a multi-wavelet.

We shall in fact show that, for any $r \in \mathbb{Z}$,

$$\mathbf{F} \in \mathbf{S}^r_{\mathbf{\Phi}} \cap \mathbf{W}^r_{\mathbf{\Phi},Q} \Rightarrow \mathbf{F} = \mathbf{0}, \text{ the zero vector function }, \tag{5.1.36}$$

which implies the desired result (5.1.35).

Suppose therefore that

$$\mathbf{F} \in \mathbf{S}^r_{\mathbf{\Phi}} \cap \mathbf{W}^r_{\mathbf{\Phi},Q} \subset \mathbf{S}^{r+1}_{\mathbf{\Phi}}, \tag{5.1.37}$$

from (5.1.2) and (5.1.10). Since (5.1.37) gives $\mathbf{F} \in \mathbf{S}^r_{\mathbf{\Phi}}$, there exists a matrix sequence $\{M_1(k)\} \in l^{\nu\times\nu}(\mathbb{Z})$ such that

$$\mathbf{F}(x) = \sum_k M_1(k)\mathbf{\Phi}(2^r x-k), \tag{5.1.38}$$

and similarly, since (5.1.37) also implies $\mathbf{F} \in \mathbf{W}^r_{\mathbf{\Phi},Q}$, there exists a matrix sequence $\{M_2(k)\} \in l^{\nu\times\nu}(\mathbb{Z})$ such that

$$\mathbf{F}(x) = \sum_k M_2(k)\mathbf{\Psi}_{\mathbf{\Phi},Q}(2^r x-k). \tag{5.1.39}$$

Now observe from (5.1.38), together with (5.1.3)-(5.1.4) in Theorem 5.1.1, that

$$\mathbf{F}(x) = \sum_k \widetilde{M}(k)\mathbf{\Phi}(2^{r+1}x-k), \tag{5.1.40}$$

where

$$\widetilde{M}(k) := \sum_j M_1(j)P(k-2j), \quad k \in \mathbb{Z}. \tag{5.1.41}$$

Similarly, according to (5.1.39), together with (5.1.11), (5.1.12) in Theorem 5.1.2, we have

$$\mathbf{F}(x) = \sum_k \widetilde{\widetilde{M}}(k)\mathbf{\Phi}(2^{r+1}x - k), \tag{5.1.42}$$

where

$$\widetilde{\widetilde{M}}(k) := \sum_j M_2(j)Q(k-2j), \quad k \in \mathbb{Z}. \tag{5.1.43}$$

Since $\mathbf{\Phi}$ possesses linearly independent integer shifts on $\mathbb{R}$, we deduce from (5.1.40) and (5.1.42) that

$$\widetilde{M}(k) = \widetilde{\widetilde{M}}(k) =: M(k), \quad k \in \mathbb{Z}. \tag{5.1.44}$$

It follows from (5.1.34), (5.1.40), (5.1.41) and (5.1.44) that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} M^{**}(k) &= \sum_j M(j)B(2k-j) = \sum_j \Big[\sum_l M_1(l)P(j-2l)\Big]B(2k-j) \\ &= \sum_l M_1(l)\Big[\sum_j P(j-2l)B(2k-j)\Big]. \end{aligned} \tag{5.1.45}$$

We proceed to show that

$$\sum_j P(j-2l)B(2k-j) = O, \quad l,k \in \mathbb{Z}, \tag{5.1.46}$$

which together with (5.1.45), will then imply that

$$M^{**}(k) = O, \quad k \in \mathbb{Z}. \tag{5.1.47}$$

To prove (5.1.46), we use the definitions in (5.1.16) and (5.1.19) to obtain, for any $l \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} &\sum_k \Big[\sum_j P(j-2l)B(2k-j)\Big]z^{2k} \\ &= \sum_k \Big[\sum_j P(2j-2l)B(2k-2j) + \sum_j P(2j+1-2l)B(2k-2j-1)\Big]z^{2k} \\ &= \sum_j P(2j-2l)\Big[\sum_k B(2k-2j)z^{2k-2j}\Big]z^{2j} + \sum_j P(2j+1-2l)\Big[\sum_k B(2k-2j-1)z^{2k-2j-1}\Big]z^{2j+1} \\ &= z^{2l}\left\{\Big[\sum_j P(2j-2l)z^{2j-2l}\Big]\Big[\sum_k B(2k)z^{2k}\Big] + \Big[\sum_j P(2j+1-2l)z^{2j+1-2l}\Big]\Big[\sum_k B(2k+1)z^{2j+1}\Big]\right\} \\ &= z^{2l}\left\{\Big[\sum_j P(2j)z^{2j}\Big]\Big[\sum_k B(2k)z^{2k}\Big] + \Big[\sum_j P(2j+1)z^{2j+1}\Big]\Big[\sum_k B(2k+1)z^{2j+1}\Big]\right\} \\ &= \frac{1}{2}z^{2l}\Big[\big(\mathcal{P}(z)+\mathcal{P}(-z)\big)\big(\mathcal{B}(z)+\mathcal{B}(-z)\big) + \big(\mathcal{P}(z)-\mathcal{P}(-z)\big)\big(\mathcal{B}(z)-\mathcal{B}(-z)\big)\Big] \\ &= z^{2l}\Big(\mathcal{P}(z)\mathcal{B}(z) + \mathcal{P}(-z)\mathcal{B}(-z)\Big) = O, \end{aligned}$$

since the identity in the third line of (5.1.21) is satisfied, and thus

$$\sum_k \Big[\sum_j P(j-2l)B(2k-j)\Big] z^{2k} = O, \quad z \in \mathbb{C} \setminus \{0\},\ l \in \mathbb{Z},$$

thereby yielding the desired result (5.1.46), and therefore also (5.1.47).

It similarly follows from (5.1.33), (5.1.42), (5.1.43) and (5.1.44) that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} M^*(k) = \sum_j M(j)A(2k-j) &= \sum_j \Big[\sum_l M_2(l)Q(j-2l)\Big] A(2k-j) \\ &= \sum_l M_2(l)\Big[\sum_j Q(j-2l)A(2k-j)\Big]. \end{aligned} \tag{5.1.48}$$

We proceed to show that

$$\sum_j Q(j-2l)A(2k-j) = O, \quad l,k \in \mathbb{Z}, \tag{5.1.49}$$

which, together with (5.1.48), will then imply that

$$M^*(k) = O, \quad k \in \mathbb{Z}. \tag{5.1.50}$$

To prove (5.1.49), we use the definitions in (5.1.17) and (5.1.18) to obtain, for any $l \in \mathbb{Z}$ and $z \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned} &\sum_k \Big[\sum_j Q(j-2l)A(2k-j)\Big] z^{2k} \\ &= \sum_k \Big[\sum_j Q(2j-2l)A(2k-2j) + \sum_j Q(2j+1-2l)A(2k-2j-1)\Big] z^{2k} \\ &= \sum_j Q(2j-2l)\Big[\sum_k A(2k-2j)z^{2k-2j}\Big] z^{2j} + \sum_j Q(2j+1-2l)\Big[\sum_k A(2k-2j-1)z^{2k-2j-1}\Big] z^{2j+1} \\ &= z^{2l}\Big\{\Big[\sum_j Q(2j-2l)z^{2j-2l}\Big]\Big[\sum_k A(2k)z^{2k}\Big] + \Big[\sum_j Q(2j+1-2l)z^{2j+1-2l}\Big]\Big[\sum_j A(2k+1)z^{2k+1}\Big]\Big\} \\ &= \frac{1}{2}z^{2l}\Big\{\Big[\mathcal{Q}(z)+\mathcal{Q}(-z)\Big]\Big[\mathcal{A}(z)+\mathcal{A}(-z)\Big] + \Big[\mathcal{Q}(z)-\mathcal{Q}(-z)\Big]\Big[\mathcal{A}(z)-\mathcal{A}(-z)\Big]\Big\} \\ &= z^{2l}\big(\mathcal{Q}(z)\mathcal{A}(z) + \mathcal{Q}(-z)\mathcal{A}(-z)\big) = O, \end{aligned}$$

from the identity in the second line of (5.1.21), and thus

$$\sum_k \Big[\sum_j Q(j-2l)A(2k-j)\Big] z^{2k} = O, \quad z \in \mathbb{C} \setminus \{0\}, \quad l \in \mathbb{Z},$$

which implies the desired result (5.1.49), and therefore also (5.1.50).

By combining (5.1.40), (5.1.42), (5.1.44), (5.1.32), (5.1.47) and (5.1.50), we deduce that $\mathbf{F} = \mathbf{0}$, the zero vector function, and thereby completing our proof. ■

The decomposition relation (5.1.32)-(5.1.34) of Theorem 5.1.4, together with the reconstruction relations (5.1.3) and (5.1.11) in, respectively, Theorems 5.1.1 and 5.1.2, yield corresponding multi-wavelet decomposition and reconstruction algorithms for vector-valued data. The (finitely-supported) matrix sequences $\{A(k)\}$ and $\{B(k)\}$ are known as, respectively, the low-pass and high-pass matrix filters corresponding to $\mathbf{\Phi}$ and its multi-wavelet $\mathbf{\Psi} = \mathbf{\Psi}_{\mathbf{\Phi},Q}$.

In multi-wavelet decomposition applications, it is desirable that, for a given refinable vector $\mathbf{\Phi}$, the matrix filters $\{A(k)\}$ and $\{B(k)\}$ be as short as possible. To achieve this end, our strategy in Sections 5.2 and 5.3 will be, for the given matrix refinement symbol $\mathcal{P}$, to obtain matrix Laurent polynomials $\mathcal{A}$ and $\mathcal{B}$ of shortest possible length satisfying the first and third identities in the system (5.1.21), before obtaining a matrix Laurent polynomial $\mathcal{Q}$ of shortest possible length satisfying the remaining second and fourth identities of the system (5.1.21) for these optimal choices of $\mathcal{A}$ and $\mathcal{B}$.

## 5.2 Hermite spline multi-wavelets

In this section, we apply Theorems 5.1.3 and 5.1.4 to obtain a Hermite spline multi-wavelet $\mathbf{\Psi} = \mathbf{\Psi}_\nu^H$ of shortest possible support corresponding to the (refinable) Hermite vector spline $\mathbf{\Phi}_\nu^H$ of Chapter 4.

Following (5.1.16), we define, for any $\nu \in \mathbb{N}$, the matrix Laurent polynomial symbol $\mathcal{P}_\nu^H$ corresponding to $\mathbf{\Phi}_\nu^H$ by

$$\mathcal{P}_\nu^H(z) := \frac{1}{2}\sum_k P_\nu^H(k) z^k, \tag{5.2.1}$$

for which it then follows from (4.1.11) and (4.3.2) that

$$\mathcal{P}_\nu^H(z) + \mathcal{P}_\nu^H(-z) = P_\nu^H(0) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2^{\nu-1}} \end{bmatrix}. \tag{5.2.2}$$

Let $\mathcal{A}_\nu^H$ be defined as the constant matrix polynomial

$$\mathcal{A}_\nu^H(z) := \left[P_\nu^H(0)\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix}, \tag{5.2.3}$$

from (5.2.2).

It then follows from (5.2.2) and (5.2.3) that

$$\mathcal{P}_\nu^H(z)\mathcal{A}_\nu^H(z) + \mathcal{P}_\nu^H(-z)\mathcal{A}_\nu^H(-z) = I, \tag{5.2.4}$$

that is, the matrix Laurent polynomial $\mathcal{A} = \mathcal{A}_\nu^H$ is a matrix polynomial of least possible degree satisfying the first identity in the system (5.1.21), with $\mathcal{P} = \mathcal{P}_\nu^H$.

Next, we observe that the definition

$$\mathcal{B}_\nu^H(z) := z^{-1}\left[\mathcal{P}_\nu^H(0)\right]^{-1} \mathcal{P}_\nu^H(-z) \tag{5.2.5}$$

yields, together with (5.2.2),

$$\begin{aligned} &\mathcal{P}_\nu^H(z)\mathcal{B}_\nu^H(z) + \mathcal{P}_\nu^H(-z)\mathcal{B}_\nu^H(-z) \\ &\quad = z^{-1}\left[\mathcal{P}_\nu^H(z)\,[\mathcal{P}_\nu^H(0)]^{-1}\left\{\mathcal{P}_\nu^H(0) - \mathcal{P}_\nu^H(z)\right\} - \left\{\mathcal{P}_\nu^H(0) - \mathcal{P}_\nu^H(z)\right\}[\mathcal{P}_\nu^H(0)]^{-1}\,\mathcal{P}_\nu^H(z)\right] \\ &\quad = z^{-1}\left[\left\{\mathcal{P}_\nu^H(z) - \mathcal{P}_\nu^H(z)\,[\mathcal{P}_\nu^H(0)]^{-1}\mathcal{P}_\nu^H(z)\right\} - \left\{\mathcal{P}_\nu^H(z) - \mathcal{P}_\nu^H(z)\,[\mathcal{P}_\nu^H(0)]^{-1}\,\mathcal{P}_\nu^H(z)\right\}\right] = O, \end{aligned}$$

that is,

$$\mathcal{P}_\nu^H(z)\mathcal{B}_\nu^H(z) + \mathcal{P}_\nu^H(-z)\mathcal{B}_\nu^H(-z) = O. \tag{5.2.6}$$

According to (5.2.5) and (4.1.11), the matrix Laurent polynomial $\mathcal{B}_\nu^H$ has length 3, and $\mathcal{B} = \mathcal{B}_\nu^H$ is therefore a matrix Laurent polynomial of shortest possible length satisfying the third identity in the system (5.1.21), with $\mathcal{P} = \mathcal{P}_\nu^H$, as follows from a proof by contradiction.

Now, define the matrix polynomial

$$\mathcal{Q}_\nu^H(z) := z\, I, \tag{5.2.7}$$

according to which

$$\mathcal{Q}_\nu^H(z) + \mathcal{Q}_\nu^H(-z) = O. \tag{5.2.8}$$

But then (5.2.3) and (5.2.8) yield, for any $z \in \mathbb{C}$,

$$\mathcal{Q}_\nu^H(z)\mathcal{A}_\nu^H(z) + \mathcal{Q}_\nu^H(-z)\mathcal{A}_\nu^H(-z) = \left\{\mathcal{Q}_\nu^H(z) + \mathcal{Q}_\nu^H(-z)\right\} \left(\mathcal{P}_\nu^H(0)\right)^{-1} = O,$$

and thus

$$\mathcal{Q}_\nu^H(z)\mathcal{A}_\nu^H(z) + \mathcal{Q}_\nu^H(-z)\mathcal{A}_\nu^H(-z) = O, \tag{5.2.9}$$

that is, the matrix polynomial $\mathcal{Q} = \mathcal{Q}_\nu^H$ satisfies the second identity in the system (5.1.21), with $\mathcal{A} = \mathcal{A}_\nu^H$. Also, $\mathcal{Q} = \mathcal{Q}_\nu^H$ is the shortest (non-zero) matrix Laurent polynomial satisfying the second identity in (5.1.21), with $\mathcal{A} = \mathcal{A}_\nu^H$.

By applying (5.2.7) and (5.2.5), and using also (5.2.2), we next deduce that, for any $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}\mathcal{Q}_\nu^H(z)\mathcal{B}_\nu^H(z) + \mathcal{Q}_\nu^H(-z)\mathcal{B}_\nu^H(-z) &= \left\{[\mathcal{P}_\nu^H(0)]^{-1}\, \mathcal{P}_\nu^H(-z) + [\mathcal{P}_\nu^H(0)]^{-1}\, \mathcal{P}_\nu^H(z)\right\} \\ &= [\mathcal{P}_\nu^H(0)]^{-1} \left\{\mathcal{P}_\nu^H(-z) + \mathcal{P}_\nu^H(z)\right\} = [\mathcal{P}_\nu^H(0)]^{-1}\, [\mathcal{P}_\nu^H(0)] = I,\end{aligned}$$

that is,

$$\mathcal{Q}_\nu^H(z)\mathcal{B}_\nu^H(z) + \mathcal{Q}_\nu^H(-z)\mathcal{B}_\nu^H(-z) = I. \tag{5.2.10}$$

According to (5.2.4), (5.2.9), (5.2.6), and (5.2.10), we have therefore now shown that the matrix Laurent polynomials $\mathcal{Q} = \mathcal{Q}_\nu^H$, $\mathcal{A} = \mathcal{A}_\nu^H$, $\mathcal{B} = \mathcal{B}_\nu^H$ satisfy the system (5.1.21), with $\mathcal{P} = \mathcal{P}_\nu^H$. Moreover, $\mathcal{Q} = \mathcal{Q}_\nu^H$, $\mathcal{A} = \mathcal{A}_\nu^H$ and $\mathcal{B} = \mathcal{B}_\nu^H$ are matrix Laurent polynomials of shortest possible length satisfying the system (5.1.21)

Following (5.1.16)-(5.1.19), we now define the matrix sequences $\left\{Q_\nu^H(k)\right\}$, $\left\{A_\nu^H(k)\right\}$ and $\left\{B_\nu^H(k)\right\}$ in $l_0^{\nu\times\nu}(\mathbb{Z})$ by

$$\sum_k Q_\nu^H(k)z^k := 2\ \mathcal{Q}_\nu^H(z); \tag{5.2.11}$$

$$\sum_k A_\nu^H(k)z^k := \mathcal{A}_\nu^H(z); \tag{5.2.12}$$

$$\sum_k B_\nu^H(k)z^k := \mathcal{B}_\nu^H(z), \tag{5.2.13}$$

and apply (5.2.3), (5.2.5), (5.2.7), as well as (5.2.2), to obtain

$$A_\nu^H(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix} \delta(k), \quad k \in \mathbb{Z}; \tag{5.2.14}$$

$$Q_\nu^H(k) = 2\, I\delta(k-1), \quad k \in \mathbb{Z}; \tag{5.2.15}$$

$$B_\nu^H(-2) = -\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix} P_\nu^H(-1); \tag{5.2.16}$$

$$B_\nu^H(-1) = I; \tag{5.2.17}$$

$$B_\nu^H(0) = -\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2^{\nu-1} \end{bmatrix} P_\nu^H(1); \tag{5.2.18}$$

$$B_\nu^H(k) = O, \quad k \in \mathbb{Z} \setminus \{-2,-1,0\}. \tag{5.2.19}$$

By substituting (4.3.3), (4.3.4) into (5.2.16) and (5.2.18), we deduce that the two matrices $B_\nu^H(-2)$ and $B_\nu^H(0)$ are given by

$$\left.\begin{aligned} \left[B_\nu^H(-2)\right]_{mn} &= \frac{(-1)^{m+n+1}}{2^{n-m}} f_{\nu,m}^{(n-1)}(1/2); \\ \left[B_\nu^H(0)\right]_{mn} &= -\frac{1}{2^{n-m}} f_{\nu,m}^{(n-1)}(1/2), \end{aligned}\right\} \quad m,n = 1,\cdots,\nu, \tag{5.2.20}$$

with $\{f_{\nu,m} : \; m = 1,\cdots,\nu\} \subset \pi_{2\nu-1}$, denoting the polynomial sequence of Theorem 4.2.1. Hence we may now apply Theorem 5.1.4, together with (5.2.1) and (5.2.14), (5.2.15), (5.2.17), (5.2.19), (5.2.20), (4.3.7), to obtain the following Hermite spline multi-wavelet result.

**Theorem 5.2.1** *For any integer $\nu \in \mathbb{N}$, let the matrix refinable Hermite vector spline $\mathbf{\Phi}_\nu^H$ and its matrix refinement sequence $\{P_\nu^H(k)\}$ be given as in Theorem 4.1.2. Then the vector spline $\mathbf{\Psi}_\nu^H = \left(\psi_{\nu,1}^H, \cdots, \psi_{\nu,\nu}^H\right)^T : \mathbb{R} \to \mathbb{R}^\nu$, as defined by*

$$\mathbf{\Psi}_\nu^H(x) := 2\mathbf{\Phi}_\nu^H(2x-1), \tag{5.2.21}$$

*is a corresponding minimally supported Hermite spline multi-wavelet, with*

$$\text{supp } \psi_{\nu,k}^H = [0,1], \quad k = 1, \cdots, \nu, \tag{5.2.22}$$

*and with decomposition relation* (5.1.32) *given, for any sequence $\{M(k)\} \in l^{\nu\times\nu}(\mathbb{Z})$, by*

$$\sum_k M(k)\mathbf{\Phi}_\nu^H(2^{r+1}x-k) = \sum_k M^*(k)\mathbf{\Phi}_\nu^H(2^r x-k) + \sum_k M^{**}(k)\mathbf{\Psi}_\nu^H(2^r x-k), \quad r \in \mathbb{Z}, \tag{5.2.23}$$

*where*

$$\left[M^*(k)\right]_{mn} = 2^{n-1}\left[M(k)\right]_{mn}, \quad m,n = 1,\cdots,\nu; \quad k \in \mathbb{Z}; \tag{5.2.24}$$

$$\begin{aligned}\left[M^{**}(k)\right]_{mn} &= \sum_{l=1}^{\nu}\left[M(2k)\right]_{ml}\frac{(-1)^n}{2^{n-l}}\ \widetilde{f}_{\nu,l}^{(n-1)}(0) + \left[M(2k+1)\right]_{mn} \\ &\quad + \sum_{l=1}^{\nu}\left[M(2k+2)\right]_{ml}\frac{(-1)^l}{2^{n-l}}\ \widetilde{f}_{\nu,l}^{(n-1)}(0), \quad m,n = 1,\cdots,\nu; \quad k \in \mathbb{Z};\end{aligned} \tag{5.2.25}$$

*with the sequences $\{\widetilde{f}_{\nu,l}^{(n-1)}(0) : \ l = 1,\cdots,\nu\}$, $n = 1,\cdots,\nu$ obtained as in Steps 1 and 2 of Algorithm 3.1.*

*Moreover, the matrix filters $\{A_\nu^H(k)\}$ and $\{B_\nu^H(k)\}$ in, respectively,* (5.2.14) *and* (5.2.16)-(5.2.19)*, and as incorporated in the decomposition relation* (5.2.23)-(5.2.25)*, are the shortest possible matrix filters corresponding to $\mathbf{\Phi}_\nu^H$.*

By using also Theorems 4.1.1 and 4.1.2, as well as Tables 4.2 and 4.3, we proceed to calculate the cases $\nu = 2$ and 3 of Theorem 5.2.1, as follows.

**Example 5.2.1** Hermite spline multiwavelets for $\nu = 2, 3$.

- $\nu = 2$

  The Hermite spline multi-wavelet $\mathbf{\Psi}_2^H = \left(\psi_{2,1}^H, \psi_{2,2}^H\right)^T$ is given by

$$\left.\begin{aligned}\psi_{2,1}^H(x) &= (-32\,x^3 + 24\,x^2)\chi_{[0,1/2)}(x) + (32\,x^3 - 72\,x^2 + 48\,x - 8)\chi_{[1/2,1)}(x) \\ \psi_{2,2}^H(x) &= (16\,x^3 - 8\,x^2)\chi_{[0,1/2)}(x) + (16\,x^3 - 40\,x^2 + 32\,x - 8)\chi_{[1/2,1)}(x)\end{aligned}\right\} \tag{5.2.26}$$

Its graph is drawn in Fig. 5.1.

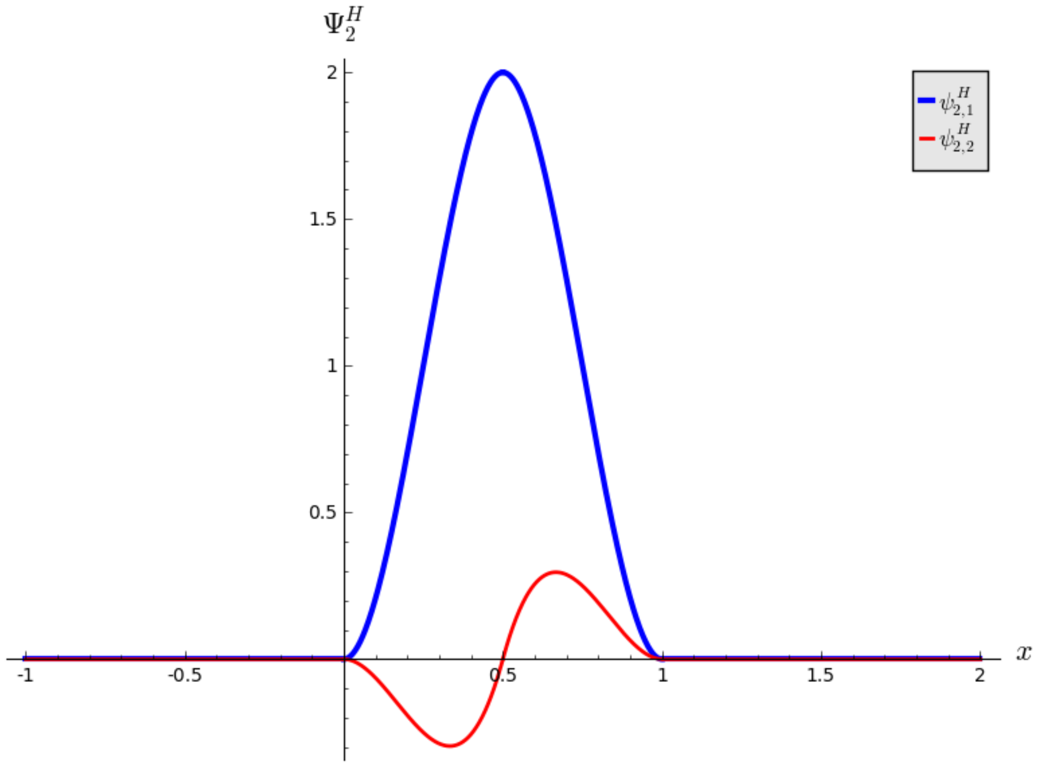



**Figure 5.1:** *Hermite spline multi-wavelet* $\mathbf{\Psi}_2^H = \left(\psi_{2,1}^H, \psi_{2,2}^H\right)^T$

- $\nu = 3$

The Hermite spline multi-wavelet $\mathbf{\Psi}_3^H = \left(\psi_{3,1}^H, \psi_{3,2}^H, \psi_{3,3}^H\right)^T$ is given by

$$\psi_{3,1}^H(x) = \begin{cases} 384\,x^5 - 480\,x^4 + 160\,x^3, & x \in [0, 1/2) \\ -384\,x^5 + 1440\,x^4 - 2080\,x^3 + 1440\,x^2 - 480\,x + 64, & x \in [1/2, 1), \end{cases} \tag{5.2.27}$$

$$\psi_{3,2}^H(x) = \begin{cases} -192\,x^5 + 224\,x^4 - 64\,x^3, & x \in [0, 1/2) \\ -192\,x^5 + 736\,x^4 - 1088\,x^3 + 768\,x^2 - 256\,x + 32, & x \in [1/2, 1), \end{cases} \tag{5.2.28}$$

$$\psi_{3,3}^H(x) = \begin{cases} 32\,x^5 - 32\,x^4 + 8\,x^3, & x \in [0, 1/2) \\ -32\,x^5 + 128\,x^4 - 200\,x^3 + 152\,x^2 - 56\,x + 8, & x \in [1/2, 1), \end{cases} \tag{5.2.29}$$

Its graph is given in Fig. 5.2.

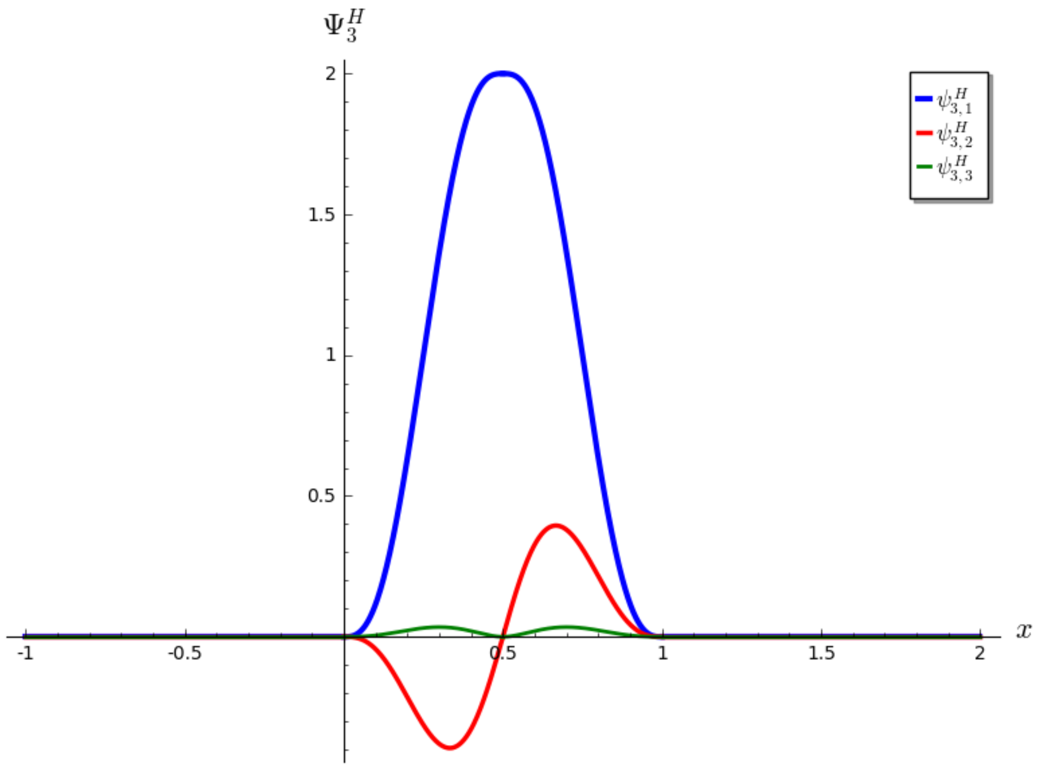



**Figure 5.2:** *Hermite spline multi-wavelet* $\mathbf{\Psi}_3^H = \left(\psi_{3,1}^H, \psi_{3,2}^H, \psi_{3,3}^H\right)^T$

■

**Remark 5.2.1** For any $\nu \in \mathbb{N}$, the graph of a Hermite spline multi-wavelet $\mathbf{\Psi}_\nu^H$ is obtained from the graph of $\mathbf{\Phi}_\nu^H$, but modified in such a way that it is compressed on the interval $[0, 1]$.

■

## 5.3 Spline multi-wavelets for $\mathbf{\Phi}_{\nu,n}$

In this section, as an extension to the vector setting of the scalar spline-wavelet results in [3, Section 9.4], we apply the general results of Section 5.1 to explicitly construct, from the refinable vector spline $\mathbf{\Phi}_{\nu,n}$ of Theorems 3.1.2 and 3.3.2, a spline multi-wavelet $\mathbf{\Psi}_{\nu,n}$, in such a manner that the corresponding matrix filters $\{A(k)\}$ and $\{B(k)\}$ are as short as possible, and with $\mathbf{\Psi}_{\nu,n}$ minimally supported for these optimal choices of $\{A(k)\}$ and $\{B(k)\}$.

To this end, we fix the integers $\nu \geq 2$ and $n \geq \rho_\nu$, as given in (3.1.30), and recall from (3.3.30) in Chapter 3 that the matrix refinement symbol $\mathcal{P}_{\nu,n}$ corresponding to $\mathbf{\Phi}_{\nu,n}$ is given, for any fixed $z \in \mathbb{C}$, by the lower triangular $\nu \times \nu$ matrix

$$\mathcal{P}_{\nu,n}(z) = \begin{bmatrix} \left(\frac{1+z}{2}\right)^{n+1} & 0 & 0 & \cdots & 0 \\ J_{n,1}(z) & (1/2)^n & 0 & \cdots & 0 \\ J_{n,2}(z) & 0 & (1/2)^{n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ J_{n,\nu-1}(z) & 0 & 0 & \cdots & (1/2)^{n+2-\nu} \end{bmatrix}, \tag{5.3.1}$$

where the polynomial sequence $\{J_{n,k} : \ k = 1, \cdots, \nu - 1\} \subset \pi_{n-1}$ is given, according to (3.3.26) and (3.3.22), by

$$J_{n,k}(z) := \frac{1}{2^{n+1}} \frac{(-1)^{k-1}\Big\{T_{n,k}(z^2) - 2^k T_{n,k}(z)\Big\}}{(1-z)^{n+1}}, \quad k = 1, \cdots, \nu - 1, \tag{5.3.2}$$

with the sequence $\{T_{n,k} : \ k = 1, \cdots, \nu - 1\} \subset \pi_n$ denoting the Taylor polynomials defined in (3.2.6), (3.2.4).

For any integer $\nu \geq 2$, let the subspace $\mathcal{S}^{\nu\times\nu}$ of $\mathcal{M}_\nu$ be defined by

$$\mathcal{S}^{\nu\times\nu} := \Big\{ M \in \mathcal{M}_\nu : \ M \text{ has, except for in its first column, zero entries off its main diagonal} \Big\},$$

that is,

$$M \in \mathcal{S}^{\nu\times\nu} \Leftrightarrow \big[M\big]_{ij} = m_{ij}, \text{ with } \left\{ \begin{array}{ll} m_{ij} := 0, & i < j; \\ m_{ij} := 0, & i > j; \quad j = 2, \cdots, \nu \quad (\text{if } \nu \geq 3). \end{array} \right\}$$

Observe from (5.3.1) that $\mathcal{P}_{\nu,n}(z) \in \mathcal{S}^{\nu\times\nu}$ for any fixed $z \in \mathbb{C} \setminus \{0\}$. Also, note that, for any $\widetilde{M}, M^* \in \mathcal{S}^{\nu\times\nu}$, the product matrix $M := \widetilde{M} M^*$ also satisfies $M \in \mathcal{S}^{\nu\times\nu}$, with, more precisely, if

$$\big[M\big]_{ij} = m_{ij}; \quad \big[\ \widetilde{M}\ \big]_{ij} = \widetilde{m}_{ij}; \quad \big[M^*\big]_{ij} = m^*_{ij},$$

then

$$\left. \begin{array}{l} m_{ii} = \widetilde{m}_{ii}\ m^*_{ii}, \quad i = 1, \cdots, \nu; \\ m_{i1} = \widetilde{m}_{i1}\ m^*_{11} + \widetilde{m}_{ii}\ m^*_{i1}, \quad i = 2, \cdots, \nu. \end{array} \right\} \tag{5.3.3}$$

We proceed to obtain matrix Laurent polynomials $\mathcal{Q}, \mathcal{A}$ and $\mathcal{B}$, with $\big\{\mathcal{Q}(z),\ \mathcal{A}(z),\ \mathcal{B}(z)\big\} \subset \mathcal{S}^{\nu\times\nu}$ for each fixed $z \in \mathbb{C} \setminus \{0\}$, and such that, with $\mathcal{P} = \mathcal{P}_{\nu,n}$, the matrix Laurent polynomial identity system (5.1.21) in Theorem 5.1.3, that is,

$$\mathcal{P}_{\nu,n}(z)\mathcal{A}(z) + \mathcal{P}_{\nu,n}(-z)\mathcal{A}(-z) = I; \tag{5.3.4}$$

$$\mathcal{Q}(z)\mathcal{A}(z) + \mathcal{Q}(-z)\mathcal{A}(-z) = O; \tag{5.3.5}$$

$$\mathcal{P}_{\nu,n}(z)\mathcal{B}(z) + \mathcal{P}_{\nu,n}(-z)\mathcal{B}(-z) = O; \tag{5.3.6}$$

$$\mathcal{Q}(z)\mathcal{B}(z) + \mathcal{Q}(-z)\mathcal{B}(-z) = I, \tag{5.3.7}$$

is satisfied.

For any fixed $z \in \mathbb{C} \setminus \{0\}$, let the $\nu \times \nu$ matrices $\left\{\mathcal{Q}(z),\ \mathcal{A}(z),\ \mathcal{B}(z)\right\} \subset \mathcal{S}^{\nu\times\nu}$ be given by

$$\left.\begin{aligned} \left[\mathcal{Q}(z)\right]_{ij} &= q_{ij}(z);\\ \left[\mathcal{A}(z)\right]_{ij} &= a_{ij}(z);\\ \left[\mathcal{B}(z)\right]_{ij} &= b_{ij}(z), \end{aligned}\right\} \quad i,j = 1,\cdots,\nu. \tag{5.3.8}$$

By applying (5.3.3), we see that the matrix Laurent polynomial system (5.3.4)-(5.3.7) is equivalent to the following system of altogether 12 identities:

$$\left(\frac{1+z}{2}\right)^{n+1} a_{11}(z) + \left(\frac{1-z}{2}\right)^{n+1} a_{11}(-z) = 1; \tag{5.3.9}$$

$$a_{kk}(z) + a_{kk}(-z) = 2^{n-k+2}, \quad k = 2,\cdots,\nu; \tag{5.3.10}$$

$$\left\{J_{n,k-1}(z)a_{11}(z) + J_{n,k-1}(-z)a_{11}(-z)\right\} + \left(\frac{1}{2}\right)^{n-k+2}\left\{a_{k1}(z) + a_{k1}(-z)\right\} = 0,$$
$$k = 2,\cdots,\nu; \tag{5.3.11}$$

$$q_{11}(z)a_{11}(z) + q_{11}(-z)a_{11}(-z) = 0; \tag{5.3.12}$$

$$q_{kk}(z)a_{kk}(z) + q_{kk}(-z)a_{kk}(-z) = 0, \quad k = 2,\cdots,\nu; \tag{5.3.13}$$

$$\left\{q_{k1}(z)a_{11}(z) + q_{k1}(-z)a_{11}(-z)\right\} + \left\{q_{kk}(z)a_{k1}(z) + q_{kk}(-z)a_{k1}(-z)\right\} = 0,$$
$$k = 2,\cdots,\nu; \tag{5.3.14}$$

$$\left(\frac{1+z}{2}\right)^{n+1} b_{11}(z) + \left(\frac{1-z}{2}\right)^{n+1} b_{11}(-z) = 0; \tag{5.3.15}$$

$$b_{kk}(z) + b_{kk}(-z) = 0, \quad k = 2,\cdots,\nu; \tag{5.3.16}$$

$$\left\{J_{n,k-1}(z)b_{11}(z) + J_{n,k-1}(-z)b_{11}(-z)\right\} + \left(\frac{1}{2}\right)^{n-k+2}\left\{b_{k1}(z) + b_{k1}(-z)\right\} = 0,$$
$$k = 2,\cdots,\nu; \tag{5.3.17}$$

$$q_{11}(z)b_{11}(z) + q_{11}(-z)b_{11}(-z) = 1; \tag{5.3.18}$$

$$q_{kk}(z)b_{kk}(z) + q_{kk}(-z)b_{kk}(-z) = 1, \quad k = 2, \cdots, \nu; \tag{5.3.19}$$

$$\{q_{k1}(z)b_{11}(z) + q_{k1}(-z)b_{11}(-z)\} + \{q_{kk}(z)b_{k1}(z) + q_{kk}(-z)b_{k1}(-z)\} = 0, \quad k = 2, \cdots, \nu. \tag{5.3.20}$$

First, to solve the identity (5.3.9), we shall rely on the following result from [3, Section 7.2].

**Theorem 5.3.1** *For any integer $n \in \mathbb{N}$, there exists precisely one polynomial $H_n$ in $\pi_{n-1}$ satisfying the identity*

$$\left(\frac{1+z}{2}\right)^{n+1} H_n(z) - \left(\frac{1-z}{2}\right)^{n+1} H_n(-z) = z^{2\lfloor (n+1)/2 \rfloor - 1}, \tag{5.3.21}$$

*where the polynomial sequence $\{H_1, H_2, \cdots\}$ satisfies the recursive formulation*

$$\left.\begin{cases} H_1(z) = 1; \\ H_{2m}(z) = \dfrac{2}{1+z}\left[H_{2m-1}(z) + (-1)^m \dbinom{2m-1}{m-1}\left(\dfrac{1-z}{2}\right)^{2m}\right]; \\ H_{2m+1}(z) = \dfrac{2}{1+z}\left[z^2 H_{2m}(z) + (-1)^m \dbinom{2m}{m}\left(\dfrac{1-z}{2}\right)^{2m+1}\right], \end{cases}\right\} \quad m = 1, 2, \cdots, \tag{5.3.22}$$

*and where, for odd values of $n$, we have the explicit formulation*

$$H_{2m-1}(z) = z^{m-1} \sum_{j=0}^{m-1} \binom{m+j-1}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j, \quad m \in \mathbb{N}. \tag{5.3.23}$$

*Also,*

$$\deg(H_n) = n - 1, \tag{5.3.24}$$

*and*

$$H_n(0) \neq 0. \tag{5.3.25}$$

With the coefficient sequence $\{h_n(j) : \ j = 0, \cdots, n-1\}$ defined by

$$\sum_{j=0}^{n-1} h_n(j) z^j := H_n(z), \tag{5.3.26}$$

the formulas in Theorem 5.3.1 yield the values in Table 5.1 of the coefficients $\{h_n(j) : \ j = 0, \cdots, n-1\}$ for $n = 1, \cdots, 5$.

*Table 5.1: Coefficients $\{h_n(j)\}$ of $H_n$, for $n = 1, \cdots, 5$*

| $n$ | $\{h_n(j) : \ j = 0, \cdots, n-1\}$ |
|---|---|
| 1 | $\{1\}$ |
| 2 | $\{\frac{3}{2}, -\frac{1}{2}\}$ |
| 3 | $\{-\frac{1}{2}, 2, -\frac{1}{2}\}$ |
| 4 | $\{-\frac{5}{8}, \frac{25}{8}, -\frac{15}{8}, \frac{3}{8}\}$ |
| 5 | $\{\frac{3}{8}, -\frac{18}{8}, \frac{38}{8}, -\frac{18}{8}, \frac{3}{8}\}$ |

Note from the uniqueness statement in Theorem 5.3.1 that $H_n$ is the polynomial of least possible degree satisfying the identity (5.3.9).

It follows from Theorem 5.3.1 that

$$a_{11}(z) = z^{-2\lfloor (n+1)/2 \rfloor + 1} H_n(z) \tag{5.3.27}$$

is a Laurent polynomial of shortest possible length satisfying the identity (5.3.9).

Next, note that the constant polynomials

$$a_{kk}(z) = 2^{n-k+1}, \quad k = 2, \cdots, \nu, \tag{5.3.28}$$

solve the identity (5.3.10). By substituting (5.3.27) into (5.3.11), we obtain the identity

$$a_{k1}(z) + a_{k1}(-z) = -2^{n-k+2} z^{-2\lfloor (n+1)/2 \rfloor + 1} \Big\{ J_{n,k-1}(z) H_n(z) - J_{n,k-1}(-z) H_n(-z) \Big\},$$
$$k = 2, \cdots, \nu. \tag{5.3.29}$$

Next, we substitute (5.3.27) into (5.3.12) to obtain

$$q_{11}(z) H_n(z) = q_{11}(-z) H_n(-z). \tag{5.3.30}$$

Now observe from (5.3.21) and (5.3.25) that the two polynomials $H_n(z)$ and $H_n(-z)$ have no common zeros in $\mathbb{C}$ and therefore also no common factors, according to which we may deduce from (5.3.30) that $q_{11}(z)$ must contain the factor $H_n(-z)$.

It follows that

$$q_{11}(z) = H_n(-z) \tag{5.3.31}$$

is a polynomial of least possible degree satisfying the identity (5.3.30). Moreover, (5.3.28) and (5.3.13) imply

$$q_{kk}(z) + q_{kk}(-z) = 0, \quad k = 2, \cdots, \nu,$$

which is solved by the polynomials

$$q_{kk}(z) = z, \quad k = 2, \cdots, \nu. \tag{5.3.32}$$

Next, we set

$$q_{k1}(z) := J_{n,k-1}(z), \quad k = 2, \cdots, \nu, \tag{5.3.33}$$

which, together with (5.3.27) and (5.3.32), we now substitute into (5.3.14) to obtain the identity

$$a_{k1}(z) - a_{k1}(-z) = -z^{-2\lfloor (n+1)/2 \rfloor}\Big\{J_{n,k-1}(z)H_n(z) - J_{n,k-1}(-z)H_n(-z)\Big\}, \quad k = 2, \cdots, \nu. \tag{5.3.34}$$

Now observe from (5.3.15) that

$$\Big(\frac{1+z}{2}\Big)^{n+1} b_{11}(z) = -\Big(\frac{1-z}{2}\Big)^{n+1} b_{11}(-z),$$

according to which $b_{11}(z)$ must contain the factor $\Big(\dfrac{1-z}{2}\Big)^{n+1}$.

It follows that a Laurent polynomial $b_{11}(z)$ of shortest possible length satisfying the identity (5.3.15) is given by

$$b_{11}(z) = -z^{-2\lfloor (n+1)/2 \rfloor + 1}\Big(\frac{1-z}{2}\Big)^{n+1}. \tag{5.3.35}$$

Also, note that the identity (5.3.16) is solved by

$$b_{kk}(z) = \frac{1}{2} z^{-1}, \quad k = 2, \cdots, \nu. \tag{5.3.36}$$

By substituting (5.3.35) into (5.3.17), we derive the identity

$$b_{k1}(z) + b_{k1}(-z) = 2^{n-k+2}\ z^{-2\lfloor (n+1)/2 \rfloor + 1}\Big\{\Big(\frac{1-z}{2}\Big)^{n+1} J_{n,k-1}(z) - \Big(\frac{1+z}{2}\Big)^{n+1} J_{n,k-1}(-z)\Big\}, \quad k = 2, \cdots, \nu. \tag{5.3.37}$$

Next, we substitute (5.3.31) and (5.3.35) into the left hand side of (5.3.18) to obtain

$$q_{11}(z)b_{11}(z) + q_{11}(-z)b_{11}(-z) = z^{-2\lfloor (n+1)/2 \rfloor + 1}\Big\{\Big(\frac{1-z}{2}\Big)^{n+1} H_n(-z) - \Big(\frac{1+z}{2}\Big)^{n+1} H_n(z)\Big\} = 1,$$

from Theorem 5.3.1, and thereby showing that the identity (5.3.18) is satisfied with $q_{11}(z)$ and $b_{11}(z)$ given as in, respectively, (5.3.31) and (5.3.35).

Now substitute (5.3.32) and (5.3.36) into the left hand side of (5.3.19) to obtain

$$q_{kk}(z)b_{kk}(z) + q_{kk}(-z)b_{kk}(-z) = z\Big(\frac{1}{2}z^{-1}\Big) + (-z)\Big(-\frac{1}{2}z^{-1}\Big) = 1,$$

and thereby showing that the identity (5.3.19) is satisfied with $q_{kk}(z)$ and $b_{kk}(z)$ given as in, respectively, (5.3.32) and (5.3.36).

Finally, by substituting (5.3.33), (5.3.35) and (5.3.32) into the left hand side of (5.3.20), we obtain

$$\begin{aligned}\big\{q_{k1}(z)b_{11}(z) + q_{k1}(-z)b_{11}(-z)\big\} &+ \big\{q_{kk}(z)b_{k1}(z) + q_{kk}(-z)b_{k1}(-z)\big\}\\ &= z^{-2\lfloor (n+1)/2\rfloor+1}\Big\{\Big(\frac{1-z}{2}\Big)^{n+1}J_{n,k-1}(z) - \Big(\frac{1+z}{2}\Big)^{n+1}J_{n,k-1}(-z)\Big\}\\ &\qquad + z\,\big\{b_{k1}(z) - b_{k1}(-z)\big\},\end{aligned}$$

from which we deduce that the identity (5.3.20) is equivalent to

$$b_{k1}(z) - b_{k1}(-z) = z^{-2\lfloor (n+1)/2\rfloor}\Big\{\Big(\frac{1-z}{2}\Big)^{n+1}J_{n,k-1}(z) - \Big(\frac{1+z}{2}\Big)^{n+1}J_{n,k-1}(-z)\Big\}. \tag{5.3.38}$$

For any fixed $k \in \{1, \cdots, \nu - 1\}$, we proceed to investigate the degrees of the two odd polynomials $U_{n,k}$ and $V_{n,k}$, as defined by

$$U_{n,k}(z) := J_{n,k}(z)H_n(z) - J_{n,k}(-z)H_n(-z); \tag{5.3.39}$$

$$V_{n,k}(z) := \Big(\frac{1+z}{2}\Big)^{n+1}J_{n,k}(-z) - \Big(\frac{1-z}{2}\Big)^{n+1}J_{n,k}(z), \tag{5.3.40}$$

and as appearing in (5.3.29), (5.3.34), (5.3.37) and (5.3.38).

Since $J_{n,k}$ and $H_n$ both belong to the polynomial space $\pi_{n-1}$, we see from (5.3.39) and (5.3.40) that the two odd polynomials $U_{n,k}$ and $V_{n,k}$ satisfy $U_{n,k} \in \pi_{2n-3}$ and $V_{n,k} \in \pi_{2n-1}$. We shall show that, in fact,

$$U_{n,k} \in \pi_{2\lfloor (n+1)/2\rfloor-3}, \quad (\text{if } n \geq 3); \quad U_{1,1} = U_{2,1} = \text{ the zero polynomial}; \tag{5.3.41}$$

$$V_{n,k} \in \pi_{2\lfloor (n+1)/2\rfloor-1}. \tag{5.3.42}$$

Our proof of (5.3.41) will depend on the following result.

**Theorem 5.3.2** *For any integer $n \in \mathbb{N}$, let $Y \in \pi_n$. Then there exists at most one polynomial pair $\{F, G\} \subset \pi_n$ satisfying the identity*

$$\left(\frac{1+z}{2}\right)^{n+1} F(z) + \left(\frac{1-z}{2}\right)^{n+1} G(z) = Y(z^2). \tag{5.3.43}$$

*If, moreover, a polynomial pair $\{F, G\} \subset \pi_n$ does indeed satisfy the identity* (5.3.43)*, then we must have*

$$F(z) = G(-z), \tag{5.3.44}$$

*and thus*

$$\left(\frac{1+z}{2}\right)^{n+1} F(z) + \left(\frac{1-z}{2}\right)^{n+1} F(-z) = Y(z^2). \tag{5.3.45}$$

**Proof.** Let $\{\widetilde{F}, \widetilde{G}\}$ and $\{F^*, G^*\}$ be two polynomial pairs in $\pi_n$ such that

$$\left(\frac{1+z}{2}\right)^{n+1} \widetilde{F}(z) + \left(\frac{1-z}{2}\right)^{n+1} \widetilde{G}(z) = Y(z^2); \tag{5.3.46}$$

$$\left(\frac{1+z}{2}\right)^{n+1} F^*(z) + \left(\frac{1-z}{2}\right)^{n+1} G^*(z) = Y(z^2). \tag{5.3.47}$$

By subtracting (5.3.47) from (5.3.46), we obtain the identity

$$\left(\frac{1+z}{2}\right)^{n+1} (\widetilde{F} - F^*)(z) = -\left(\frac{1-z}{2}\right)^{n+1} (\widetilde{G} - G^*)(z). \tag{5.3.48}$$

It follows from (5.3.48) that

$$(\widetilde{F} - F^*)(z) = \left(\frac{1-z}{2}\right)^{n+1} K(z), \tag{5.3.49}$$

for some polynomial $K$. But $\widetilde{F}, F^* \in \pi_n$ implies that $\widetilde{F} - F^* \in \pi_n$, which, together with (5.3.49), yields $K$=the zero polynomial, and thus also $\widetilde{F} - F^*$=the zero polynomial. According to (5.3.48), we then also have $\widetilde{G} - G^*$=the zero polynomial. Hence $\widetilde{F} = F^*$, $\widetilde{G} = G^*$, which then concludes the proof of the uniqueness part of the theorem.

Suppose next $\{F, G\} \subset \pi_n$ is a polynomial pair in $\pi_n$ satisfying the identity (5.3.43). By replacing $z$ by $-z$ in (5.3.43), we obtain

$$\left(\frac{1+z}{2}\right)^{n+1} G(-z) + \left(\frac{1-z}{2}\right)^{n+1} F(-z) = Y(z^2). \tag{5.3.50}$$

Since $\{F, G\} \subset \pi_n$, both the polynomials $F(-z)$ and $G(-z)$ also belong to the space $\pi_n$, so that we apply the uniqueness statement of the theorem to deduce from (5.3.43) and (5.3.50) that

$$F(z) = G(-z); \quad G(z) = F(-z), \tag{5.3.51}$$

which proves (5.3.44).

Finally, observe that the second identity in (5.3.44), together with (5.3.43), yields the identity (5.3.45). ■

Now let us apply Theorem 5.3.2 to prove (5.3.41).

To this end, we first replace $z$ by $-z$ in (5.3.2) to obtain

$$J_{n,k}(-z) = \frac{(-1)^{k-1}}{2^{n+1}(1+z)^{n+1}}\Big\{T_{n,k}(z^2) - 2^k T_{n,k}(-z)\Big\}. \tag{5.3.52}$$

It then follows from (5.3.2) and (5.3.52) that the polynomial $J_{n,k}$ satisfies the identity

$$\left(\frac{1+z}{2}\right)^{n+1} J_{n,k}(-z) - \left(\frac{1-z}{2}\right)^{n+1} J_{n,k}(z) = zW_{n,k}(z^2), \tag{5.3.53}$$

where the polynomial $W_{n,k}$ is defined by

$$W_{n,k}(z^2) := \frac{(-1)^{k+1}}{2^{2n+2-k}}\, z^{-1} \Big\{T_{n,k}(z) - T_{n,k}(-z)\Big\}. \tag{5.3.54}$$

Let the two polynomial pairs $\{\widetilde{Q}_{n,k}, \widetilde{R}_{n,k}\}$ and $\{Q^*_{n,k}, R^*_{n,k}\}$, with $\widetilde{R}_{n,k}, R^*_{n,k} \in \pi_n$, be determined by the polynomial division procedures

$$\left.\begin{aligned} zH_n(-z)W_{n,k}(z^2) &= \left(\frac{1+z}{2}\right)^{n+1} \widetilde{Q}_{n,k}(z) + \widetilde{R}_{n,k}(z); \\ z^{2\lfloor (n+1)/2 \rfloor - 1} J_{n,k}(z) &= \left(\frac{1+z}{2}\right)^{n+1} Q^*_{n,k}(z) + R^*_{n,k}(z). \end{aligned}\right\} \tag{5.3.55}$$

It then follows from (5.3.21), (5.3.53) and (5.3.55) that

$$\left.\begin{aligned} \left(\frac{1+z}{2}\right)^{n+1} \widetilde{X}_{n,k}(z) + \left(\frac{1-z}{2}\right)^{n+1}\Big\{-\widetilde{R}_{n,k}(z)\Big\} &= z^{2\lfloor (n+1)/2\rfloor} W_{n,k}(z^2); \\ \left(\frac{1+z}{2}\right)^{n+1} X^*_{n,k}(z) + \left(\frac{1-z}{2}\right)^{n+1}\Big\{-R^*_{n,k}(z)\Big\} &= z^{2\lfloor (n+1)/2\rfloor} W_{n,k}(z^2), \end{aligned}\right\} \tag{5.3.56}$$

where

$$\left.\begin{aligned} \widetilde{X}_{n,k}(z) &:= zH_n(z)W_{n,k}(z^2) - \left(\frac{1-z}{2}\right)^{n+1} \widetilde{Q}_{n,k}(z); \\ X^*_{n,k}(z) &:= z^{2\lfloor (n+1)/2\rfloor - 1} J_{n,k}(-z) - \left(\frac{1-z}{2}\right)^{n+1} Q^*_{n,k}(z). \end{aligned}\right\} \tag{5.3.57}$$

According to (5.3.54) and $T_{n,k} \in \pi_n$, the right-hand side of the identities (5.3.56) belongs to the polynomial space $\pi_{2n}$. Since, moreover, $\widetilde{R}_{n,k}, R^*_{n,k} \in \pi_n$, we deduce from (5.3.56) that also $\widetilde{X}_{n,k}, X^*_{n,k} \in \pi_n$. Hence we may apply Theorem 5.3.2 to deduce from (5.3.56) that

$$\widetilde{X}_{n,k} = X^*_{n,k} =: X_{n,k}; \quad \widetilde{R}_{n,k} = R^*_{n,k} =: R_{n,k}, \tag{5.3.58}$$

and thus also

$$X_{n,k}(z) = -R_{n,k}(-z). \tag{5.3.59}$$

Hence we may now subtract the identities in (5.3.55), as well as the definitions in (5.3.57), and apply (5.3.58), to obtain the two identities

$$\left.\begin{aligned} zH_n(-z)W_{n,k}(z^2) - z^{2\lfloor (n+1)/2\rfloor -1}J_{n,k}(z) &= \left(\frac{1+z}{2}\right)^{n+1}\Big\{\widetilde{Q}_{n,k}(z) - Q^*_{n,k}(z)\Big\};\\ zH_n(z)W_{n,k}(z^2) - z^{2\lfloor (n+1)/2\rfloor -1}J_{n,k}(-z) &= \left(\frac{1-z}{2}\right)^{n+1}\Big\{\widetilde{Q}_{n,k}(z) - Q^*_{n,k}(z)\Big\}.\end{aligned}\right\} \tag{5.3.60}$$

Now multiply the two identities in (5.3.60) by, respectively, $J_{n,k}(-z)$ and $J_{n,k}(z)$, and subtract the resulting two identities, to deduce the identity

$$\begin{aligned} &zW_{n,k}(z^2)\Big\{J_{n,k}(z)H_n(z) - J_{n,k}(-z)H_n(-z)\Big\}\\ &\quad = \left\{\left(\frac{1+z}{2}\right)^{n+1}J_{n,k}(-z) - \left(\frac{1-z}{2}\right)^{n+1}J_{n,k}(z)\right\}\Big\{Q^*_{n,k}(z) - \widetilde{Q}_{n,k}(z)\Big\}\\ &\quad = zW_{n,k}(z^2)\Big\{Q^*_{n,k}(z) - \widetilde{Q}_{n,k}(z)\Big\},\end{aligned}$$

by virtue of (5.3.53), and thus

$$J_{n,k}(z)H_n(z) - J_{n,k}(-z)H_n(-z) = Q^*_{n,k}(z) - \widetilde{Q}_{n,k}(z). \tag{5.3.61}$$

To investigate the degree of the polynomial $Q^* - \widetilde{Q}$ in (5.3.61), we first observe from $H_n \in \pi_{n-1}$, $J_{n,k} \in \pi_{n-1}$, together with (5.3.54) and $T_{n,k} \in \pi_n$, that the first identity in (5.3.60) has a left-hand side of degree at most $2n-1$ and a right-hand side of degree equal to $n+1+\deg(Q^* - \widetilde{Q})$, according to which $Q^*_{n,k} - \widetilde{Q}_{n,k} \in \pi_{n-2}$, which, together with (5.3.39) and (5.3.61), establishes the desired result (5.3.41).

For $n = 2$ and $k = 1$, we deduce from (5.3.39), together with Tables 3.3 and 5.1, that $U_{1,1} = U_{2,1} =$ the zero polynomial, which completes our proof of (5.3.41).

Observe that the result (5.3.42) is an immediate consequence of (5.3.40), (5.3.53), (5.3.54) and $T_{n,k} \in \pi_{n-1}$.

Let the coefficient sequences $\{\sigma_{n,k}(l) : \ l = 0, \cdots, 2n-2\}$ and $\{\tau_{n,k}(l) : \ l = 0, \cdots, n\}$ be defined by

$$\sum_{l=0}^{2n-2} \sigma_{n,k}(l)z^l := J_{n,k}(z)H_n(z), \tag{5.3.62}$$

and, as before in (3.2.25),

$$\sum_{l=0}^{n} \tau_{n,k}(l)z^l := T_{n,k}(z). \tag{5.3.63}$$

It then follows from (5.3.62) and (5.3.63), together with (5.3.39)-(5.3.42), that

$$J_{n,k}(z)H_n(z) - J_{n,k}(-z)H_n(-z) = \begin{cases} \text{the zero polynomial}, \quad k=1, \quad \text{if } n \in \{1,2\}; \\ 2 \displaystyle\sum_{l=0}^{\lfloor (n+1)/2 \rfloor -2} \sigma_{n,k}(2l+1)z^{2l+1}, \\ \qquad\qquad k=1,\cdots,\nu-1, \quad \text{if } n \geq 3; \end{cases} \tag{5.3.64}$$

$$\left(\frac{1+z}{2}\right)^{n+1} J_{n,k}(-z) - \left(\frac{1-z}{2}\right)^{n+1} J_{n,k}(z) = \frac{(-1)^{k+1}}{2^{2n-k+1}} \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} \tau_{n,k}(2l+1)z^{2l+1},$$
$$k=1,\cdots,\nu-1. \tag{5.3.65}$$

By substituting (5.3.64) into (5.3.29) and (5.3.34), we obtain

$$a_{k1}(z) + a_{k1}(-z) = \begin{cases} \text{the zero polynomial}, \quad k=2, \quad \text{if } n \in \{1,2\}; \\ -2^{n-k+3} z^{-2\lfloor (n+1)/2 \rfloor +1} \displaystyle\sum_{l=0}^{\lfloor (n+1)/2 \rfloor -2} \sigma_{n,k-1}(2l+1)z^{2l+1}, \\ \qquad\qquad k=2,\cdots,\nu, \quad \text{if } n \geq 3; \end{cases} \tag{5.3.66}$$

$$a_{k1}(z) - a_{k1}(-z) = \begin{cases} \text{the zero polynomial}, \quad k=2, \quad \text{if } n \in \{1,2\}; \\ -2\ z^{-2\lfloor (n+1)/2 \rfloor} \displaystyle\sum_{l=0}^{\lfloor (n+1)/2 \rfloor -2} \sigma_{n,k-1}(2l+1)z^{2l+1}, \\ \qquad\qquad k=2,\cdots,\nu, \quad \text{if } n \geq 3. \end{cases} \tag{5.3.67}$$

By adding the two identities (5.3.66) and (5.3.67), we obtain

$$a_{k1}(z) = \begin{cases} \text{the zero polynomial}, \quad k=2, \quad \text{if } n \in \{1,2\}; \\ -z^{-2\lfloor (n+1)/2 \rfloor}(1+2^{n-k+2}z) \displaystyle\sum_{l=0}^{\lfloor (n+1)/2 \rfloor -2} \sigma_{n,k-1}(2l+1)z^{2l+1}, \\ \qquad\qquad k=2,\cdots,\nu, \quad \text{if } n \geq 3. \end{cases} \tag{5.3.68}$$

Observe from (5.3.62), (3.3.28) and (5.3.26) that, for any $k \in \{1,\cdots,\nu-1\}$, and with the definitions

$$\gamma_{n,k}(m) := 0, \quad m \in \mathbb{Z} \setminus \{0,\cdots,n-1\};$$

$$h_n(j) := 0, \quad j \in \mathbb{Z} \setminus \{0,\cdots,n-1\},$$

we have

$$\begin{aligned}\sum_{l=0}^{2n-2} \sigma_{n,k}(l) z^l &= \Big\{ \sum_m \gamma_{n,k}(m)\ z^m \Big\} \Big\{ \sum_j h_n(j)\ z^j \Big\} \\ &= \sum_m \gamma_{n,k}(m) \Big\{ \sum_j h_n(j)\ z^{j+m} \Big\} \\ &= \sum_m \gamma_{n,k}(m) \Big\{ \sum_j h_n(j-m)\ z^j \Big\} \\ &= \sum_j \Big\{ \sum_m \gamma_{n,k}(m)\ h_n(j-m) \Big\}\ z^j \\ &= \sum_{l=0}^{2n-2} \Big\{ \sum_{m=0}^{l} \gamma_{n,k}(m)\ h_n(l-m) \Big\}\ z^l,\end{aligned}$$

and thus,

$$\sigma_{n,k}(l) = \sum_{m=0}^{l} \gamma_{n,k}(m)\ h_n(l-m), \quad l = 0, \cdots, 2n-2, \tag{5.3.69}$$

according to which

$$\sigma_{n,k-1}(2l+1) = \sum_{l=0}^{2l+1} \gamma_{n,k}(m)\ h_n(2l+1-m), \quad l = 0, \cdots, \lfloor (n+1)/2 \rfloor - 2. \tag{5.3.70}$$

Calculating by means of (5.3.70), for $n = 3, 4, 5$, together with Tables 3.3 and 5.1, we obtain the values in Table 5.2 for the coefficient sequence $\{\sigma_{n,k-1}(2l+1) :\ l = 0, \cdots, \lfloor (n+1)/2 \rfloor - 2\}$ in the second line of (5.3.68) for $n = 3, 4, 5$.

*Table 5.2: Coefficients $\{\sigma_{n,k-1}(2l+1) :\ l = 0, \cdots, \lfloor (n+1)/2 \rfloor - 2\}$ for $n = 3, 4, 5$*

| $n$ | $k$ | $\{\sigma_{n,k-1}(2l+1) :\ l = 0, \cdots, \lfloor (n+1)/2 \rfloor - 2\}$ |
|---|---|---|
| 3 | 2 | $\left\{ \frac{3}{16} \right\}$ |
| 3 | 3 | $\left\{ \frac{5}{8} \right\}$ |
| 4 | 2 | $\left\{ \frac{5}{32} \right\}$ |
| 4 | 3 | $\left\{ \frac{65}{96} \right\}$ |
| 5 | 2 | $\left\{ -\frac{15}{256}, \frac{25}{256} \right\}$ |
| 5 | 3 | $\left\{ -\frac{77}{256}, \frac{305}{768} \right\}$ |

Next, by substituting (5.3.65) into (5.3.37) and (5.3.38), we obtain

$$b_{k1}(z) + b_{k1}(-z) = \frac{(-1)^{k+1}}{2^n}\, z^{-2\lfloor (n+1)/2 \rfloor + 1} \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} \tau_{n,k-1}(2l+1) z^{2l+1},$$

$$k = 2, \cdots, \nu; \qquad (5.3.71)$$

$$b_{k1}(z) - b_{k1}(-z) = \frac{(-1)^{k+1}}{2^{2n-k+2}}\, z^{-2\lfloor (n+1)/2 \rfloor} \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} \tau_{n,k-1}(2l+1) z^{2l+1},$$

$$k = 2, \cdots, \nu. \qquad (5.3.72)$$

By adding the two identities (5.3.71) and (5.3.72), we obtain

$$b_{k1}(z) = (-1)^{k+1}\, z^{-2\lfloor (n+1)/2 \rfloor} \left( \frac{z}{2^{n+1}} + \frac{1}{2^{2n-k+3}} \right) \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} \tau_{n,k-1}(2l+1) z^{2l+1},$$

$$k = 2, \cdots, \nu. \qquad (5.3.73)$$

By using Table 3.2, we obtain the values in Table 5.3 for the coefficient sequence $\{\tau_{n,k-1}(2l+1) :\ l = 0, \cdots, \lfloor (n-1)/2 \rfloor\}$ for $n \in \{1, \cdots, 5\}$.

*Table 5.3: Coefficients* $\{\tau_{n,k-1}(2l+1) :\ l = 0, \cdots, \lfloor (n-1)/2 \rfloor\}$ *for* $n = 1, \cdots, 5$

| $n$ | $k$ | $\{\tau_{n,k-1}(2l+1) :\ l = 0, \cdots, \lfloor (n-1)/2 \rfloor\}$ |
|---|---|---|
| 1 | 2 | $\{1\}$ |
| 2 | 2 | $\{2\}$ |
| 2 | 3 | $\{-2\}$ |
| 3 | 2 | $\{3, \frac{1}{3}\}$ |
| 3 | 3 | $\{-5, -1\}$ |
| 4 | 2 | $\{4, \frac{4}{3}\}$ |
| 4 | 3 | $\{-\frac{26}{3}, -\frac{14}{3}\}$ |
| 5 | 2 | $\{5, \frac{10}{3}, \frac{1}{5}\}$ |
| 5 | 3 | $\{-\frac{77}{6}, -13, -, \frac{5}{6}\}$ |

By combining (5.3.27), (5.3.28), (5.3.31)-(5.3.33), (5.3.35), (5.3.36), (5.3.68), (5.3.73), (5.3.62)-(5.3.63), as well as (5.3.41) and (5.3.42), we can therefore now state the following result.

**Theorem 5.3.3** *For any integers* $\nu \geq 2$ *and* $n \geq \rho_\nu$, *with* $\rho_\nu$ *defined by* (3.1.30), *and with the matrix polynomial* $\mathcal{P}_{\nu,n}$ *defined by* (5.3.1), *let* $\mathcal{Q}_{\nu,n}$ *be the matrix polynomial, and*

*$\mathcal{A}_{\nu,n}$, $\mathcal{B}_{\nu,n}$ the matrix Laurent polynomials, with $\mathcal{A}_{\nu,n}(z)$, $\mathcal{Q}_{\nu,n}(z)$, $\mathcal{B}_{\nu,n}(z) \in \mathcal{S}^{\nu\times\nu}$ for each fixed $z \in \mathbb{C}\setminus\{0\}$, as given in the notation*

$$\left.\begin{aligned} \big[\mathcal{A}_{\nu,n}(z)\big]_{ij} &= a_{ij}^{\nu,n}(z);\\ \big[\mathcal{Q}_{\nu,n}(z)\big]_{ij} &= q_{ij}^{\nu,n}(z);\\ \big[\mathcal{B}_{\nu,n}(z)\big]_{ij} &= b_{ij}^{\nu,n}(z); \end{aligned}\right\} \quad i,j = 1,\cdots,\nu, \tag{5.3.74}$$

*by*

$$a_{11}^{\nu,n}(z) := z^{-2\lfloor (n+1)/2\rfloor+1} H_n(z); \tag{5.3.75}$$

$$a_{kk}^{\nu,n}(z) := 2^{n-k+1}, \quad k = 2,\cdots,\nu; \tag{5.3.76}$$

$$a_{k1}^{\nu,n}(z) := \begin{cases} \text{the zero polynomial }, \quad k=2, \quad \textit{if } n\in\{1,2\};\\ -z^{-2\lfloor (n+1)/2\rfloor}(1+2^{n-k+2}z)\displaystyle\sum_{l=0}^{\lfloor (n+1)/2\rfloor-2}\sigma_{n,k-1}(2l+1)z^{2l+1},\\ \qquad\qquad\qquad\qquad k=2,\cdots,\nu, \quad \textit{if } n\geq 3; \end{cases} \tag{5.3.77}$$

$$q_{11}^{\nu,n}(z) := H_n(-z); \tag{5.3.78}$$

$$q_{kk}^{\nu,n}(z) := z, \quad k = 2,\cdots,\nu; \tag{5.3.79}$$

$$q_{k1}^{\nu,n}(z) := J_{n,k-1}(z), \quad k = 2,\cdots,\nu; \tag{5.3.80}$$

$$b_{11}^{\nu,n}(z) := z^{-2\lfloor (n+1)/2\rfloor+1}\left(\frac{1-z}{2}\right)^{n+1}; \tag{5.3.81}$$

$$b_{kk}^{\nu,n}(z) := \frac{1}{2}z^{-1}, \quad k = 2,\cdots,\nu; \tag{5.3.82}$$

$$b_{k1}^{\nu,n}(z) := (-1)^{k+1}z^{-2\lfloor (n+1)/2\rfloor}\left(\frac{1}{2^{2n-k+3}}+\frac{z}{2^{n+1}}\right)\sum_{l=0}^{\lfloor (n-1)/2\rfloor}\tau_{n,k-1}(2l+1)z^{2l+1},$$
$$k = 2,\cdots,\nu. \tag{5.3.83}$$

*with $H_n \in \pi_{n-1}$ as in Theorem 5.3.1, where $J_{n,k}$ is the polynomial defined by (5.3.2) and where the coefficient sequences $\{\sigma_{n,k}(2l+1):\ l = 0,\cdots,\lfloor (n+1)/2\rfloor - 2\}$, if $n\geq 3$, and $\{\tau_{n,k}(2l+1):\ l=0,\cdots,\lfloor (n-1)/2\rfloor\}$ are obtained from the definitions*

$$\left.\begin{aligned} \sum_{l=0}^{2n-2}\sigma_{n,k}(l)z^l &:= J_{n,k}(z)H_n(z);\\ \sum_{l=0}^{n}\tau_{n,k}(l)z^l &:= T_{n,k}(z), \end{aligned}\right\} \quad k = 1,\cdots,\nu-1, \tag{5.3.84}$$

*with $T_{n,k} \in \pi_n$ denoting the Taylor polynomial defined in* (3.2.6), (3.2.4). *Then the matrix Laurent polynomials $\mathcal{P}_{\nu,n}$ , $\mathcal{A}_{\nu,n}$ , $\mathcal{Q}_{\nu,n}$ and $\mathcal{B}_{\nu,n}$ satisfy the system*

$$\left.\begin{aligned} \mathcal{P}_{\nu,n}(z)\mathcal{A}_{\nu,n}(z) + \mathcal{P}_{\nu,n}(-z)\mathcal{A}_{\nu,n}(-z) = I; \\ \mathcal{Q}_{\nu,n}(z)\mathcal{A}_{\nu,n}(z) + \mathcal{Q}_{\nu,n}(-z)\mathcal{A}_{\nu,n}(-z) = O; \\ \mathcal{P}_{\nu,n}(z)\mathcal{B}_{\nu,n}(z) + \mathcal{P}_{\nu,n}(-z)\mathcal{B}_{\nu,n}(-z) = O; \\ \mathcal{Q}_{\nu,n}(z)\mathcal{B}_{\nu,n}(z) + \mathcal{Q}_{\nu,n}(-z)\mathcal{B}_{\nu,n}(-z) = I. \end{aligned}\right\} \tag{5.3.85}$$

We now let the matrix sequences $\Big\{\{P_{\nu,n}(k)\},\ \{Q_{\nu,n}(k)\},\ \{A_{\nu,n}(k)\},\ \{B_{\nu,n}(k)\}\Big\} \subset l_0^{\nu\times\nu}(\mathbb{Z})$ be defined, according to (5.1.16)-(5.1.19), by

$$\sum_k P_{\nu,n}(k)z^k := 2\mathcal{P}_{\nu,n}(z); \tag{5.3.86}$$

$$\sum_k Q_{\nu,n}(k)z^k := 2\mathcal{Q}_{\nu,n}(z); \tag{5.3.87}$$

$$\sum_k A_{\nu,n}(k)z^k := \mathcal{A}_{\nu,n}(z); \tag{5.3.88}$$

$$\sum_k B_{\nu,n}(k)z^k := \mathcal{B}_{\nu,n}(z), \tag{5.3.89}$$

with the matrix polynomials $\mathcal{P}_{\nu,n}$ , $\mathcal{Q}_{\nu,n}$ and the matrix Laurent polynomials $\mathcal{A}_{\nu,n}$ , $\mathcal{B}_{\nu,n}$ given as in Theorem 5.3.3, and according to which then, with $\sigma_n$ defined by (3.1.1),

$$P_{\nu,n}(k) = O, \quad k \notin \{0, \cdots, n+1\}; \tag{5.3.90}$$

$$Q_{\nu,n}(k) = O, \quad k \notin \{0, \cdots, \sigma_n + 1\}; \tag{5.3.91}$$

$$A_{\nu,n}(k) = O, \quad k \notin \{-2\lfloor (n+1)/2 \rfloor + 1, \cdots, \sigma_n - (-2\lfloor (n+1)/2 \rfloor + 1)\}; \tag{5.3.92}$$

$$B_{\nu,n}(k) = O, \quad k \notin \{-2\lfloor (n+1)/2 \rfloor + 1, \cdots, n+1 - (-2\lfloor (n+1)/2 \rfloor + 1)\}. \tag{5.3.93}$$

Let the length of any matrix sequence $\{M(k)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$ be defined by

$$\sharp\{M(k)\} := \max\{k : \ M(k) \neq O\ \} - \min\{k : \ M(k) \neq O\ \} + 1. \tag{5.3.94}$$

It then follows from (5.3.90)-(5.3.93), together with (5.3.24) and (5.3.25) in Theorem 5.3.1, that

$$\sharp\{P_{\nu,n}(k)\} = n + 2; \tag{5.3.95}$$

$$\sharp\{Q_{\nu,n}(k)\} = \sigma_n + 1; \tag{5.3.96}$$

$$\sharp\{A_{\nu,n}(k)\} = \sigma_n + 1; \tag{5.3.97}$$

$$\sharp\{B_{\nu,n}(k)\} = n + 2. \tag{5.3.98}$$

According to our construction procedure, the respective low-pass and high-pass filter sequence lengths in (5.3.97) and (5.3.98) are the shortest possible for the refinable vector spline $\mathbf{\Phi}_{\nu,n}$.

Since also, from (5.3.85) and Theorem 5.1.3, the identity

$$\mathcal{B}_{\nu,n}(-z)\mathcal{Q}_{\nu,n}(z) = -\mathcal{A}_{\nu,n}(-z)\mathcal{P}_{\nu,n}(z) \tag{5.3.99}$$

is satisfied, we deduce from (5.3.86)-(5.3.93) that the length of the sequence $\{Q_{\nu,n}(k)\}$ must then be given by (5.3.96).

Let the coefficient sequence $\{h_n(l) : \ l = 0, \cdots, n-1\}$ be defined as in (5.3.26), with

$$\sum_{l=0}^{n-1} h_n(l)z^l := H_n(z), \tag{5.3.100}$$

whereas, for any $k \in \{1, \cdots, \nu\}$, the coefficient sequence $\{\gamma_{n,k}(l) : \ l = 0, \cdots, n-1\}$ is defined, as in (3.3.28), by

$$\sum_{l=0}^{n-1} \gamma_{n,k}(l)z^l := J_{n,k}(z). \tag{5.3.101}$$

It then follows from (5.3.87), together with (5.3.78)-(5.3.80), and $Q_{\nu,n}(l) \in \mathcal{S}^{\nu\times\nu}, \ l \in \mathbb{Z}$, that

$$Q_{\nu,n}(l) = 2\begin{bmatrix} (-1)^l h_n(l) & 0 & 0 & \cdots & 0 \\ \gamma_{n,1}(l) & \delta(l-1) & 0 & \cdots & 0 \\ \gamma_{n,2}(l) & 0 & \delta(l-1) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{n,\nu-1}(l) & 0 & 0 & \cdots & \delta(l-1) \end{bmatrix}, \quad l = 0, \cdots, \sigma_n. \tag{5.3.102}$$

According to the definition in Theorem 5.1.4, we then have

$$\mathbf{\Psi}_{\nu,n} := \mathbf{\Psi}_{\mathbf{\Phi}_{\nu,n},\{Q_{\nu,n}(k)\}} = \left(\psi_{\nu,n,1}, \cdots, \psi_{\nu,n,\nu}\right)^T, \tag{5.3.103}$$

$$\text{with} \quad \begin{bmatrix} \psi_{\nu,n,1}(x) \\ \vdots \\ \psi_{\nu,n,\nu}(x) \end{bmatrix} = \sum_{l=0}^{\sigma_n} Q_{\nu,n}(l) \begin{bmatrix} \phi_{\nu,n,1}(2x-l) \\ \vdots \\ \phi_{\nu,n,\nu}(2x-l) \end{bmatrix}, \tag{5.3.104}$$

where, from (3.1.34),

$$\phi_{\nu,n,1} := N_n; \quad \phi_{\nu,n,l} := G_{n,l-1}, \quad l = 2, \cdots, \nu, \tag{5.3.105}$$

and with the spline sequence $\{G_{n,k} : k = 1, \cdots, \nu - 1\}$ given as in Theorem 3.1.1. It follows from (5.3.104) and (5.3.102) that the component splines of the vector spline $\mathbf{\Psi}_{\nu,n} = (\psi_{\nu,n,1}, \cdots, \psi_{\nu,n,\nu})^T$ are given by

$$\psi_{\nu,n,1}(x) = 2\sum_{l=0}^{\sigma_n}(-1)^l\ h_n(l)\ N_n(2x-l); \tag{5.3.106}$$

$$\psi_{\nu,n,k}(x) = 2\ \sum_{l=0}^{\sigma_n}\gamma_{n,k-1}(l)\ N_n(2x-l) + 2\ G_{n,k-1}(2x-1), \quad k = 2, \cdots, \nu. \tag{5.3.107}$$

Observe from (5.3.106), (5.3.107), (3.1.21), as well as $\operatorname{supp} N_n = [0, n+1]$, together with the fact that Table 5.1 gives $h_1(1) = 0$, that

$$\operatorname{supp}\psi_{\nu,n,k} = [0, n], \quad k = 1, \cdots, \nu. \tag{5.3.108}$$

By using also Theorem 5.1.4, we have therefore now established the following main result of this section.

**Theorem 5.3.4** *For any integers $\nu \geq 2$ and $n \geq \rho_\nu$, with $\rho_\nu$ given by* (3.1.30)*, let $\mathbf{\Phi}_{\nu,n} = (N_n, G_{n,1}, \cdots, G_{n,\nu-1})^T$ denote the refinable vector spline of Theorem* 3.3.2*, and let the vector spline $\mathbf{\Psi}_{\nu,n} = (\psi_{\nu,n,1}, \cdots, \psi_{\nu,n,\nu})^T$ be defined by* (5.3.106), (5.3.107)*, with the coefficient sequences $\{h_n(l) : l = 0, \cdots, n-1\}$ and $\{\gamma_{n,k-1}(l) : l = 0, \cdots, n-1, \quad k = 2, \cdots, \nu\}$ defined as in* (5.3.100) *and* (5.3.101)*. Then $\mathbf{\Psi}_{\nu,n}$ is a spline multi-wavelet corresponding to $\mathbf{\Phi}_{\nu,n}$, with support interval of its component splines given by* (5.3.108)*, and with decomposition relation given, for any matrix sequence $\{M(k)\} \in l^{\nu\times\nu}(\mathbb{Z})$, by*

$$\sum_k M(k)\mathbf{\Phi}_{\nu,n}(2^{r+1}x-k) = \sum_k M^*(k)\mathbf{\Phi}_{\nu,n}(2^r x-k) + \sum_k M^{**}(k)\mathbf{\Psi}_{\nu,n}(2^r x-k), \quad r \in \mathbb{Z}, \tag{5.3.109}$$

*where*

$$M^*(k) := \sum_j M(j)A_{\nu,n}(2k-j); \tag{5.3.110}$$

$$M^{**}(k) := \sum_j M(j)B_{\nu,n}(2k-j), \tag{5.3.111}$$

*with $\{A_{\nu,n}(k)\}$, $\{B_{\nu,n}(k)\} \in l_0^{\nu\times\nu}(\mathbb{Z})$ denoting, respectively, the corresponding low-pass and high-pass matrix filters, as defined in* (5.3.88), (5.3.89)*, together with* (5.3.75)-(5.3.77) *and* (5.3.81)-(5.3.83)*. Also, $\{A_{\nu,n}(k)\}$ and $\{B_{\nu,n}(k)\}$ are the shortest possible matrix filter sequences corresponding to $\mathbf{\Phi}_{\nu,n}$, and $\mathbf{\Psi}_{\nu,n}$ is a minimally supported multi-wavelet corresponding to these optimal matrix filters $\{A_{\nu,n}(k)\}$ and $\{B_{\nu,n}(k)\}$.*

**Remark 5.3.1** Note that, for any $n \in \mathbb{N}$, the (scalar) wavelet construction in [3, Section 9.2] may be obtained by setting $\nu = 1$, and $\rho_1 = 1$, in Theorem 5.3.4. ■

We proceed to calculate the cases $\nu = 2,\ n = 1, 2, 3$; as well as $\nu = 3,\ n = 3, 4, 5$ of Theorem 5.3.3. To obtain the spline multi-wavelets $\mathbf{\Psi}_{\nu,n}$, we apply (5.3.106) and (5.3.107), together with Tables 5.1, 3.3, and 3.1, as well as the definition in (1.3.3)-(1.3.4).

- $\nu = 2,\ n = 1$

  The spline multi-wavelet $\mathbf{\Psi}_{2,1} = \left(\psi_{2,1,1}, \psi_{2,1,2}\right)^T$ is given by

$$\psi_{2,1,1}(x) := \begin{cases} 4\,x & x \in [0, 1/2) \\ -4\,x + 4 & x \in [1/2, 1) \\ 0, & x \in \mathbb{R} \setminus [0, 1), \end{cases} \tag{5.3.112}$$

$$\psi_{2,1,2}(x) := \begin{cases} x, & x \in [0, 1/2) \\ -5\,x + 5, & x \in [1/2, 1) \\ 0, & x \in \mathbb{R} \setminus [0, 1). \end{cases} \tag{5.3.113}$$

  The graph of $\mathbf{\Psi}_{2,1}$ is given in Fig. 5.3.

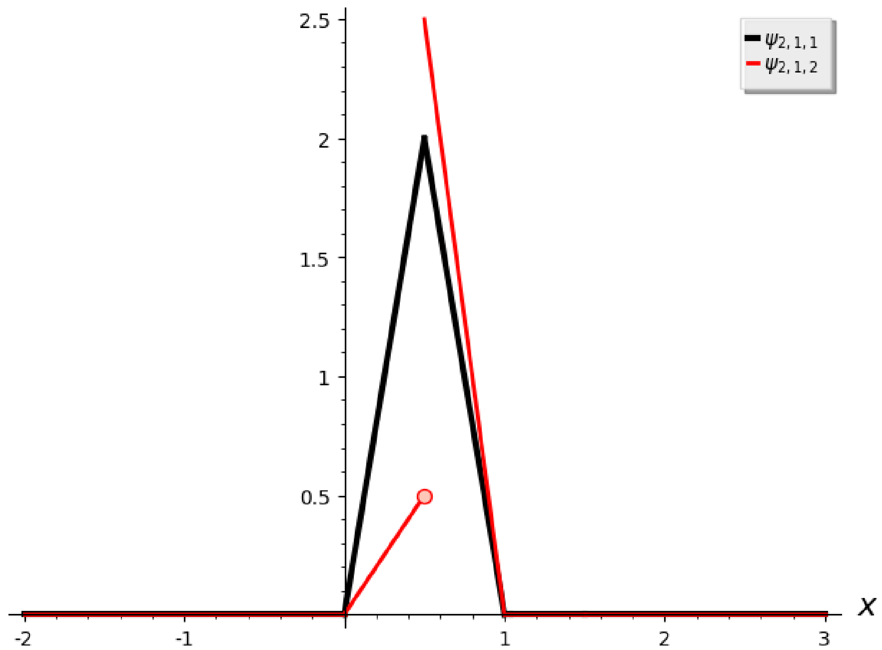



**Figure 5.3:** *Spline multi-wavelet* $\mathbf{\Psi}_{2,1} = \left(\psi_{2,1,1}, \psi_{2,1,2}\right)^T$

- $\nu = 2,\ n = 2$

The spline multi-wavelet $\mathbf{\Psi}_{2,2} = \left(\psi_{2,2,1}, \psi_{2,2,2}\right)^T$ is given by

$$\psi_{2,2,1}(x) := \begin{cases} 6\,x^2 & \text{on } (0, 1/2) \\ -10\,x^2 + 16\,x - 4, & x \in [1/2, 1) \\ 2\,x^2 - 8\,x + 8, & x \in [1, 2) \\ 0, & x \in \mathbb{R} \setminus [0, 2). \end{cases} \tag{5.3.114}$$

$$\psi_{2,2,2}(x) := \begin{cases} \frac{3}{4}\,x^2, & x \in [0, 1/2) \\ -\frac{29}{4}\,x^2 + 12\,x - 4, & x \in [1/2, 1) \\ \frac{9}{4}\,x^2 - 7\,x + \frac{11}{2}, & x \in [1, 3/2) \\ \frac{1}{4}\,x^2 - x + 1, & x \in [3/2, 2) \\ 0, & x \in \mathbb{R} \setminus [0, 2). \end{cases} \tag{5.3.115}$$

The graph of $\mathbf{\Psi}_{2,2}$ is given in Fig. 5.4.

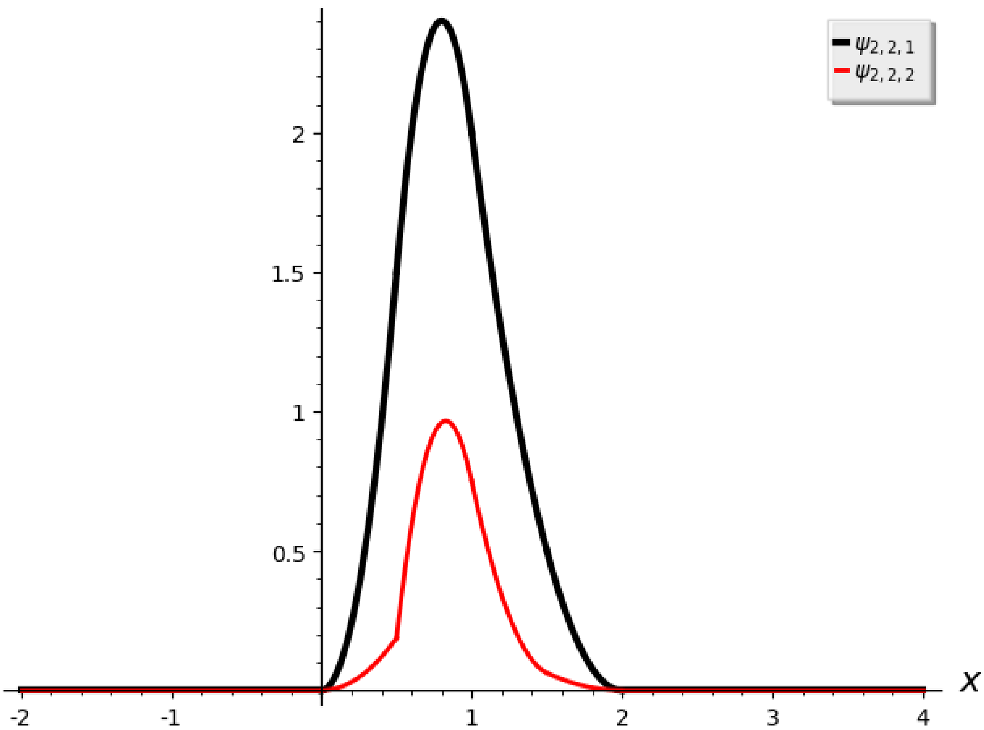



**Figure 5.4:** *Spline multi-wavelet* $\mathbf{\Psi}_{2,2} = \left(\psi_{2,2,1}, \psi_{2,2,2}\right)^T$

- $\nu = 2,\ n = 3$

The spline multi-wavelet $\boldsymbol{\Psi}_{2,3} = \left(\psi_{2,3,1}, \psi_{2,3,2}\right)^T$ is given by

$$\psi_{2,3,1}(x) := \begin{cases} -\frac{4}{3}\,x^3, & x \in [0,1) \\ \frac{32}{3}\,x^3 - 36\,x^2 + 36\,x - 12, & x \in [1,3/2) \\ -\frac{32}{3}\,x^3 + 60\,x^2 - 108\,x + 60, & x \in [3/2,2) \\ \frac{4}{3}\,x^3 - 12\,x^2 + 36\,x - 36, & x \in [2,3) \\ 0, & x \in \mathbb{R} \setminus [0,3). \end{cases} \tag{5.3.116}$$

$$\psi_{2,3,2}(x) := \begin{cases} \frac{11}{36}\,x^3, & x \in [0,1/2) \\ -\frac{67}{12}\,x^3 + \frac{77}{6}\,x^2 - \frac{101}{12}\,x + \frac{125}{72}, & x \in [1/2,1) \\ \frac{41}{12}\,x^3 - \frac{85}{6}\,x^2 + \frac{223}{12}\,x - \frac{523}{72}, & x \in [1,3/2) \\ -\frac{25}{36}\,x^3 + \frac{13}{3}\,x^2 - \frac{55}{6}\,x + \frac{119}{18}, & x \in [3/2,2) \\ -\frac{1}{18}\,x^3 + \frac{1}{2}\,x^2 - \frac{3}{2}\,x + \frac{3}{2}, & x \in [2,3) \\ 0, & x \in \mathbb{R} \setminus [0,3). \end{cases} \tag{5.3.117}$$

The graph of $\boldsymbol{\Psi}_{2,3}$ is given in Fig. 5.5.

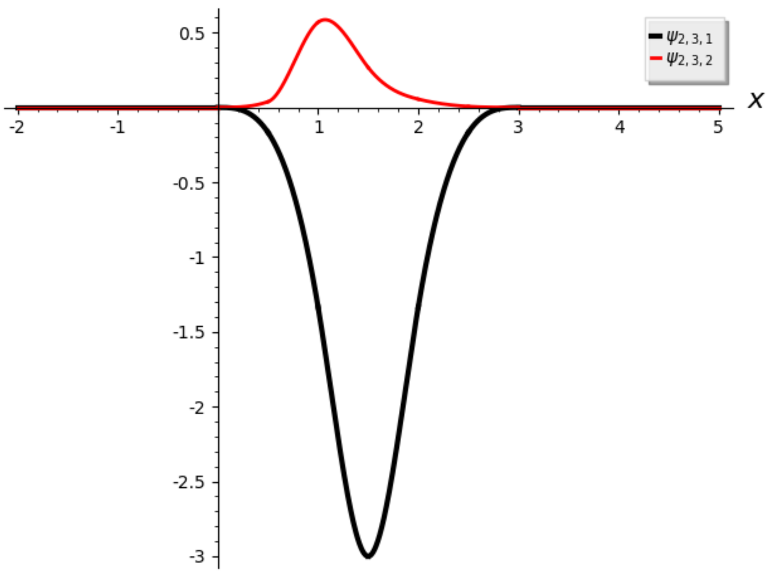



**Figure 5.5:** *Spline multi-wavelet* $\boldsymbol{\Psi}_{2,3} = \left(\psi_{2,3,1}, \psi_{2,3,2}\right)^T$

- $\nu = 3,\ n = 3$

The spline multi-wavelet $\boldsymbol{\Psi}_{3,3} = \left(\psi_{3,3,1}, \psi_{3,3,2}, \psi_{3,3,3}\right)^T$ is given by

$$\psi_{3,3,1}(x) := \begin{cases} -\frac{4}{3}\,x^3 & x \in [0,1) \\ \frac{32}{3}\,x^3 - 36\,x^2 + 36\,x - 12, & x \in [1,3/2) \\ -\frac{32}{3}\,x^3 + 60\,x^2 - 108\,x + 60, & x \in [3/2,2) \\ \frac{4}{3}\,x^3 - 12\,x^2 + 36\,x - 36, & x \in [2,5/2) \\ \frac{4}{3}\,x^3 - 12\,x^2 + 36\,x - 36, & x \in [5/2,3) \\ 0, & x \in \mathbb{R} \setminus [0,3), \end{cases} \tag{5.3.118}$$

$$\psi_{3,3,2}(x) := \begin{cases} \frac{11}{36}\,x^3 & x \in [0,1/2) \\ -\frac{67}{12}\,x^3 + \frac{77}{6}\,x^2 - \frac{101}{12}\,x + \frac{125}{72}, & x \in [1/2,1) \\ \frac{41}{12}\,x^3 - \frac{85}{6}\,x^2 + \frac{223}{12}\,x - \frac{523}{72}, & x \in [1,3/2) \\ -\frac{25}{36}\,x^3 + \frac{13}{3}\,x^2 - \frac{55}{6}\,x + \frac{119}{18}, & x \in [3/2,2) \\ -\frac{1}{18}\,x^3 + \frac{1}{2}\,x^2 - \frac{3}{2}\,x + \frac{3}{2}, & x \in [2,3) \\ 0, & x \in \mathbb{R} \setminus [0,3), \end{cases} \tag{5.3.119}$$

$$\psi_{3,3,3}(x) := \begin{cases} x^3, & x \in [0,1/2) \\ -\frac{23}{3}\,x^3 + 13\,x^2 - \frac{5}{2}\,x - \frac{11}{12}, & x \in [1/2,1) \\ \frac{55}{6}\,x^3 - \frac{75}{2}\,x^2 + 48\,x - \frac{71}{4}, & x \in [1,3/2) \\ -\frac{13}{6}\,x^3 + \frac{27}{2}\,x^2 - \frac{57}{2}\,x + \frac{41}{2}, & x \in [3/2,2) \\ -\frac{1}{6}\,x^3 + \frac{3}{2}\,x^2 - \frac{9}{2}\,x + \frac{9}{2}, & x \in [2,3) \\ 0, & x \in \mathbb{R} \setminus [0,3). \end{cases} \tag{5.3.120}$$

The graph of $\boldsymbol{\Psi}_{3,3}$ is given in Fig. 5.6.

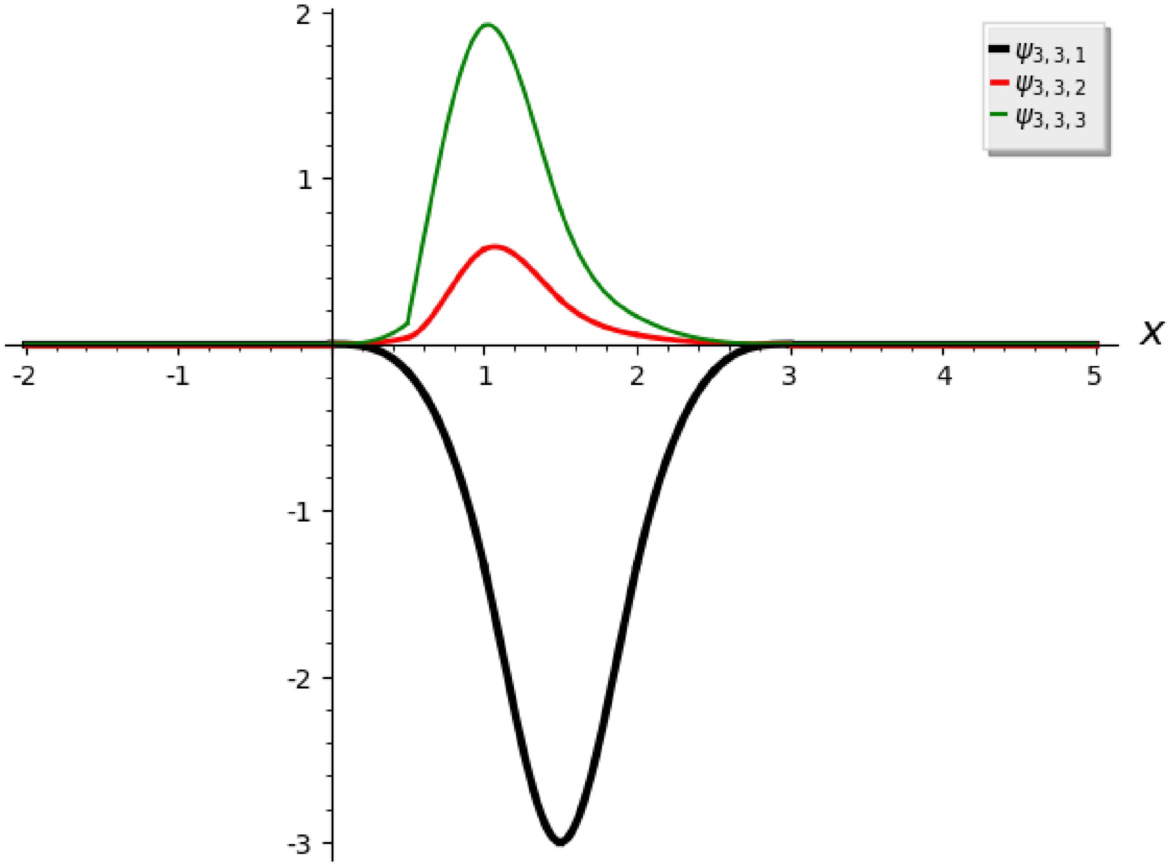



**Figure 5.6:** *Spline multi-wavelet* $\mathbf{\Psi}_{3,3} = \left(\psi_{3,3,1}, \psi_{3,3,2}, \psi_{3,3,3}\right)^T$

- $\nu = 3,\ n = 4$

The spline multi-wavelet $\mathbf{\Psi}_{3,4} = \left(\psi_{3,4,1}, \psi_{3,4,2}, \psi_{3,4,3}\right)^T$ is given by

$$\psi_{3,4,1}(x) := \begin{cases} -\frac{5}{6}\,x^4, & x \in [0,1) \\ \frac{55}{6}\,x^4 - 40\,x^3 + 60\,x^2 - 40\,x + 10, & x \in [1, 3/2) \\ -\frac{73}{6}\,x^4 + 88\,x^3 - 228\,x^2 + 248\,x - 98, & x \in [3/2, 2) \\ \frac{17}{6}\,x^4 - 32\,x^3 + 132\,x^2 - 232\,x + 142, & x \in [2,3) \\ -\frac{1}{2}\,x^4 + 8\,x^3 - 48\,x^2 + 128\,x - 128 & x \in [3,4) \\ 0, & x \in \mathbb{R} \setminus [0,4), \end{cases} \tag{5.3.121}$$

$$\psi_{3,4,2}(x) := \begin{cases} \frac{25}{288}\,x^4, & x \in [0,1/2) \\ -\frac{871}{288}\,x^4 + \frac{80}{9}\,x^3 - \frac{26}{3}\,x^2 + \frac{32}{9}\,x - \frac{19}{36}, & x \in [1/2,1) \\ \frac{785}{288}\,x^4 - \frac{127}{9}\,x^3 + \frac{155}{6}\,x^2 - \frac{175}{9}\,x + \frac{47}{9}, & x \in [1,3/2) \\ -\frac{133}{96}\,x^4 + \frac{95}{9}\,x^3 - \frac{89}{3}\,x^2 + \frac{649}{18}\,x - \frac{2245}{144}, & x \in [3/2,2) \\ \frac{83}{288}\,x^4 - \frac{17}{6}\,x^3 + \frac{21}{2}\,x^2 - \frac{35}{2}\,x + \frac{179}{16}, & x \in [2,5/2) \\ -\frac{13}{288}\,x^4 + \frac{1}{2}\,x^3 - 2\,x^2 + \frac{10}{3}\,x - \frac{11}{6}, & x \in [5/2,3) \\ \frac{1}{96}\,x^4 - \frac{1}{6}\,x^3 + x^2 - \frac{8}{3}\,x + \frac{8}{3}, & x \in [3,4) \\ 0, & x \in \mathbb{R} \setminus [0,4), \end{cases} \tag{5.3.122}$$

$$\psi_{3,4,3}(x) := \begin{cases} \frac{35}{96}\,x^4, & x \in [0,1/2) \\ -\frac{159}{32}\,x^4 + \frac{32}{3}\,x^3 - 4\,x^2 - \frac{4}{3}\,x + \frac{2}{3}, & x \in [1/2,1) \\ \frac{273}{32}\,x^4 - \frac{130}{3}\,x^3 + 77\,x^2 - \frac{166}{3}\,x + \frac{85}{6}, & x \in [1,3/2) \\ -\frac{1415}{288}\,x^4 + \frac{112}{3}\,x^3 - \frac{209}{2}\,x^2 + \frac{757}{6}\,x - \frac{2587}{48}, & x \in [3/2,2) \\ \frac{307}{288}\,x^4 - \frac{21}{2}\,x^3 + 39\,x^2 - \frac{391}{6}\,x + \frac{2005}{48}, & x \in [2,5/2) \\ -\frac{5}{32}\,x^4 + \frac{31}{18}\,x^3 - \frac{41}{6}\,x^2 + \frac{101}{9}\,x - \frac{215}{36}, & x \in [5/2,3) \\ \frac{11}{288}\,x^4 - \frac{11}{18}\,x^3 + \frac{11}{3}\,x^2 - \frac{88}{9}\,x + \frac{88}{9}, & x \in [3,4 \\ 0, & x \in \mathbb{R} \setminus [0,4). \end{cases} \tag{5.3.123}$$

The graph of $\boldsymbol{\Psi}_{3,4}$ is given in Fig. 5.7.

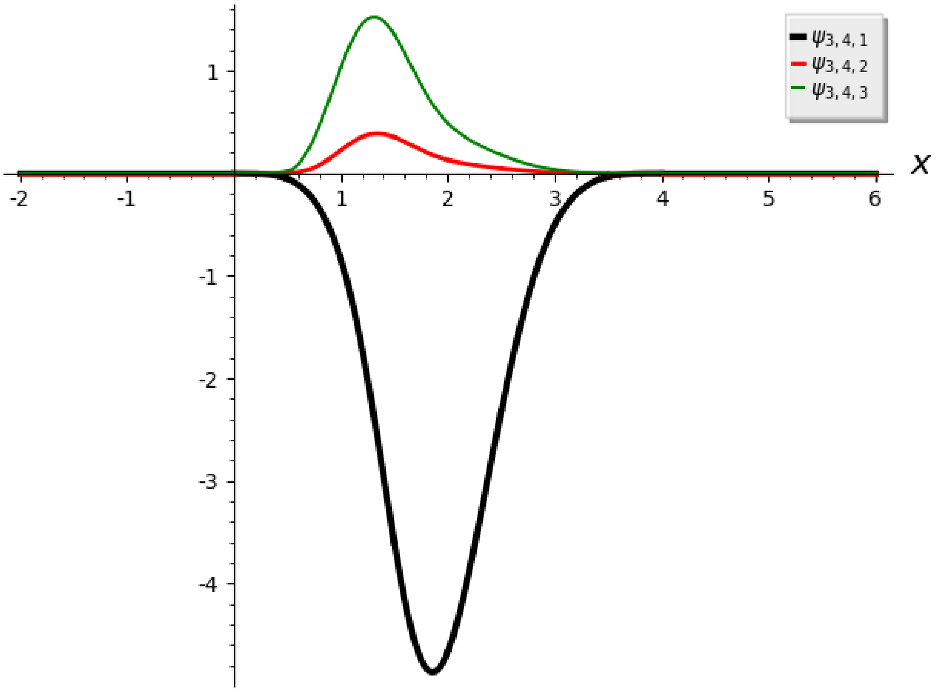



**Figure 5.7:** *Spline multi-wavelet* $\mathbf{\Psi}_{3,4} = \left(\psi_{3,4,1}, \psi_{3,4,2}, \psi_{3,4,3}\right)^T$

- $\nu = 3, \;\; n = 5$

  The spline multi-wavelet $\mathbf{\Psi}_{3,5} = \left(\psi_{3,5,1}, \psi_{3,5,2}, \psi_{3,5,3}\right)^T$ is given by

$$\psi_{3,5,1}(x) := \begin{cases} \frac{1}{5}\, x^5, & x \in [0,1) \\ -\frac{22}{15}\, x^5 + \frac{25}{3}\, x^4 - \frac{50}{3}\, x^3 + \frac{50}{3}\, x^2 - \frac{25}{3}\, x + \frac{5}{3}, & x \in [1,2) \\ \frac{128}{15}\, x^5 - \frac{275}{3}\, x^4 + \frac{1150}{3}\, x^3 - \frac{2350}{3}\, x^2 + \frac{2375}{3}\, x - \frac{955}{3}, & x \in [2,5/2) \\ -\frac{128}{15}\, x^5 + \frac{365}{3}\, x^4 - \frac{2050}{3}\, x^3 + \frac{5650}{3}\, x^2 - \frac{7625}{3}\, x + \frac{4045}{3}, & x \in [5/2,3) \\ \frac{22}{15}\, x^5 - \frac{85}{3}\, x^4 + \frac{650}{3}\, x^3 - \frac{2450}{3}\, x^2 + \frac{4525}{3}\, x - \frac{3245}{3}, & x \in [3,4) \\ -\frac{1}{5}\, x^5 + 5\, x^4 - 50\, x^3 + 250\, x^2 - 625\, x + 625, & x \in [4,5) \\ 0, & x \in \mathbb{R} \setminus [0,5), \end{cases} \tag{5.3.124}$$

$$\psi_{3,5,2}(x) := \begin{cases} \frac{137}{7200}\,x^5, & x \in [0,1/2) \\ -\frac{3077}{2400}\,x^5 + \frac{1651}{360}\,x^4 - \frac{2131}{360}\,x^3 + \frac{2611}{720}\,x^2 - \frac{3091}{2880}\,x + \frac{3571}{28800}, & x \in [1/2,1) \\ \frac{3623}{2400}\,x^5 - \frac{1687}{180}\,x^4 + \frac{7919}{360}\,x^3 - \frac{17489}{720}\,x^2 + \frac{37109}{2880}\,x - \frac{76829}{28800}, & x \in [1,3/2) \\ -\frac{8731}{7200}\,x^5 + \frac{497}{45}\,x^4 - \frac{14131}{360}\,x^3 + \frac{48661}{720}\,x^2 - \frac{161341}{2880}\,x + \frac{518521}{28800}, & x \in [3/2,2) \\ \frac{3919}{7200}\,x^5 - \frac{261}{40}\,x^4 + \frac{1241}{40}\,x^3 - \frac{17513}{240}\,x^2 + \frac{27051}{320}\,x - \frac{366893}{9600}, & x \in [2,5/2) \\ -\frac{181}{1440}\,x^5 + \frac{37}{20}\,x^4 - \frac{217}{20}\,x^3 + \frac{1903}{60}\,x^2 - \frac{1853}{40}\,x + \frac{16327}{600}, & x \in [5/2,3) \\ \frac{7}{800}\,x^5 - \frac{1}{6}\,x^4 + \frac{5}{4}\,x^3 - \frac{55}{12}\,x^2 + \frac{65}{8}\,x - \frac{131}{24}, & x \in [3,4) \\ -\frac{1}{600}\,x^5 + \frac{1}{24}\,x^4 - \frac{5}{12}\,x^3 + \frac{25}{12}\,x^2 - \frac{125}{24}\,x + \frac{125}{24}, & \text{on } (4,5) \\ 0, & x \in \mathbb{R} \setminus [0,5), \end{cases} \tag{5.3.125}$$

$$\psi_{3,5,3}(x) := \begin{cases} \frac{3}{32}\,x^5, & x \in [0,1/2) \\ -\frac{3361}{1440}\,x^5 + \frac{437}{72}\,x^4 - \frac{245}{72}\,x^3 - \frac{139}{144}\,x^2 + \frac{715}{576}\,x - \frac{1483}{5760}, & x \in [1/2,1) \\ \frac{1501}{288}\,x^5 - \frac{4559}{144}\,x^4 + \frac{1297}{18}\,x^3 - \frac{11005}{144}\,x^2 + \frac{22447}{576}\,x - \frac{44947}{5760}, & x \in [1,3/2) \\ -\frac{1363}{288}\,x^5 + \frac{6181}{144}\,x^4 - \frac{5461}{36}\,x^3 + \frac{37325}{144}\,x^2 - \frac{122543}{576}\,x + \frac{390023}{5760}, & x \in [3/2,2) \\ \frac{3199}{1440}\,x^5 - \frac{3833}{144}\,x^4 + \frac{4553}{36}\,x^3 - \frac{42787}{144}\,x^2 + \frac{197905}{576}\,x - \frac{891769}{5760}, & x \in [2,5/2) \\ -\frac{149}{288}\,x^5 + \frac{1097}{144}\,x^4 - \frac{1073}{24}\,x^3 + \frac{9419}{72}\,x^2 - \frac{9185}{48}\,x + \frac{81107}{720}, & x \in [5/2,3) \\ \frac{17}{480}\,x^5 - \frac{97}{144}\,x^4 + \frac{121}{24}\,x^3 - \frac{1327}{72}\,x^2 + \frac{1561}{48}\,x - \frac{15607}{720}, & x \in [3,4) \\ -\frac{1}{144}\,x^5 + \frac{25}{144}\,x^4 - \frac{125}{72}\,x^3 + \frac{625}{72}\,x^2 - \frac{3125}{144}\,x + \frac{3125}{144}, & x \in [4,5) \\ 0, & x \in \mathbb{R} \setminus [0,5). \end{cases} \tag{5.3.126}$$

The graph of $\mathbf{\Psi}_{3,5}$ is given in Fig. 5.8.

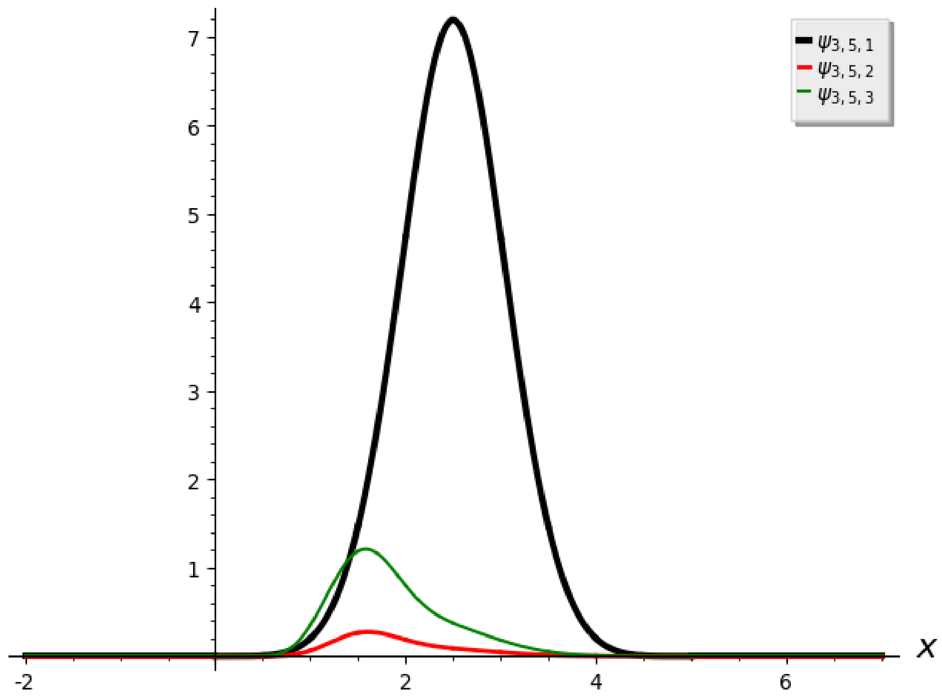



**Figure 5.8:** *Spline multi-wavelet* $\boldsymbol{\Psi}_{3,5} = \left(\psi_{3,5,1}, \psi_{3,5,2}, \psi_{3,5,3}\right)^T$

Finally, in Tables 5.4-5.5, the matrix filters $\{A_{\nu,n}(k)\}$ and $\{B_{\nu,n}(k)\}$, as obtained from Theorem 5.3.3 and Tables 5.1-5.3, together with Table 3.2 or 3.3, are given for some values of $\nu$ and $n$.

*Table 5.4: Matrix filter* $\{A_{\nu,n}(k)\}$ *for* $\nu = 2,\ n = 1,2,3$ *and* $\nu = 3,\ n = 3,4,5$

| $\nu$ | $n$ | $\{A_{\nu,n}(k)\}$ |
|---|---|---|
| 2 | 1 | $A_{2,1}(-1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2,1}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{2,1}(k) = O,\ k \in \mathbb{Z} \setminus \{-1, 0\}.$ |
| 2 | 2 | $A_{2,2}(-1) = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2,2}(0) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}, \quad A_{2,2}(k) = O,\ k \in \mathbb{Z} \setminus \{-1, 0\}.$ |

| $\nu$ | $n$ | $\{A_{\nu,n}(k)\}$ |
|---|---|---|
| 2 | 3 | $A_{2,3}(-3) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{16} & 0 \end{bmatrix}, \quad A_{2,3}(-2) = \begin{bmatrix} 2 & 0 \\ -\frac{3}{2} & 0 \end{bmatrix}, \quad A_{2,3}(-1) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix},$ $A_{2,3}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \quad A_{2,3}(k) = O,\ k \in \mathbb{Z} \setminus \{-3, \cdots, 0\}.$ |
| 3 | 3 | $A_{3,3}(-3) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{3}{16} & 0 & 0 \\ -\frac{5}{8} & 0 & 0 \end{bmatrix}, \quad A_{3,3}(-2) = \begin{bmatrix} 2 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \\ -\frac{5}{2} & 0 & 0 \end{bmatrix}, \quad A_{3,3}(-1) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$ $A_{3,3}(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_{3,3}(k) = O,\ k \in \mathbb{Z} \setminus \{-3, \cdots, 0\}.$ |
| 3 | 4 | $A_{3,4}(-3) = \begin{bmatrix} -\frac{5}{8} & 0 & 0 \\ -\frac{5}{32} & 0 & 0 \\ -\frac{65}{96} & 0 & 0 \end{bmatrix}, \quad A_{3,4}(-2) = \begin{bmatrix} \frac{25}{8} & 0 & 0 \\ -\frac{5}{2} & 0 & 0 \\ -\frac{65}{12} & 0 & 0 \end{bmatrix}, \quad A_{3,4}(-1) = \begin{bmatrix} -\frac{15}{8} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$ $A_{3,4}(0) = \begin{bmatrix} \frac{3}{8} & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A_{3,4}(k) = O,\ k \in \mathbb{Z} \setminus \{-3, \cdots, 0\}.$ |
| 3 | 5 | $A_{3,5}(-5) = \begin{bmatrix} \frac{3}{8} & 0 & 0 \\ \frac{15}{256} & 0 & 0 \\ \frac{77}{256} & 0 & 0 \end{bmatrix}, \quad A_{3,5}(-4) = \begin{bmatrix} -\frac{9}{4} & 0 & 0 \\ \frac{15}{8} & 0 & 0 \\ \frac{77}{16} & 0 & 0 \end{bmatrix}, \quad A_{3,5}(-3) = \begin{bmatrix} \frac{19}{4} & 0 & 0 \\ -\frac{25}{256} & 0 & 0 \\ -\frac{305}{768} & 0 & 0 \end{bmatrix},$ $A_{3,5}(-2) = \begin{bmatrix} -\frac{9}{4} & 0 & 0 \\ -\frac{25}{8} & 0 & 0 \\ -\frac{305}{48} & 0 & 0 \end{bmatrix}, \quad A_{3,5}(-1) = \begin{bmatrix} \frac{3}{8} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{3,5}(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 8 \end{bmatrix},$ $A_{3,5}(k) = O,\ k \in \mathbb{Z} \setminus \{-5, \cdots, 0\}.$ |

*Table 5.5: Matrix filter $\{B_{\nu,n}(k)\}$ for $\nu = 2,\ n = 1,2,3$ and $\nu = 3,\ n = 3,4,5$*

| $\nu$ | $n$ | $\{B_{\nu,n}(k)\}$ |
|---|---|---|
| 2 | 1 | $B_{2,1}(-1) = \begin{bmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix},\ B_{2,1}(0) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix},\ B_{2,1}(1) = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix},$ $B_{2,1}(k) = O,\ k \in \mathbb{Z} \setminus \{-1,0,1\}.$ |
| 2 | 2 | $B_{2,2}(-1) = \begin{bmatrix} -\frac{1}{8} & 0 \\ -\frac{1}{16} & \frac{1}{2} \end{bmatrix},\ B_{2,2}(0) = \begin{bmatrix} \frac{3}{8} & 0 \\ -\frac{1}{4} & 0 \end{bmatrix},\ B_{2,2}(1) = \begin{bmatrix} -\frac{3}{8} & 0 \\ 0 & 0 \end{bmatrix},$ $B_{2,2}(2) = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & 0 \end{bmatrix},\ B_{2,2}(k) = O,\ k \in \mathbb{Z} \setminus \{-1,0,2\}.$ |
| 2 | 3 | $B_{2,3}(-3) = \begin{bmatrix} -\frac{1}{16} & 0 \\ -\frac{3}{128} & 0 \end{bmatrix},\ B_{2,3}(-2) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{16} & 0 \end{bmatrix},\ B_{2,3}(-1) = \begin{bmatrix} -\frac{3}{8} & 0 \\ -\frac{1}{384} & \frac{1}{2} \end{bmatrix},$ $B_{2,3}(0) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{48} & 0 \end{bmatrix},\ B_{2,3}(1) = \begin{bmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{bmatrix},\ B_{2,3}(k) = O,\ k \in \mathbb{Z} \setminus \{-3,\cdots,1\}.$ |
| 3 | 3 | $B_{3,3}(-3) = \begin{bmatrix} -\frac{1}{16} & 0 & 0 \\ -\frac{3}{128} & 0 & 0 \\ -\frac{5}{64} & 0 & 0 \end{bmatrix},\ B_{3,3}(-2) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{3}{16} & 0 & 0 \\ -\frac{5}{16} & 0 & 0 \end{bmatrix},$ $B_{3,3}(-1) = \begin{bmatrix} -\frac{3}{8} & 0 & 0 \\ -\frac{1}{384} & \frac{1}{2} & 0 \\ -\frac{1}{64} & 0 & \frac{1}{2} \end{bmatrix},\ B_{3,3}(0) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{48} & 0 & 0 \\ -\frac{1}{16} & 0 & 0 \end{bmatrix},$ $B_{3,3}(1) = \begin{bmatrix} -\frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B_{3,3}(k) = O,\ k \in \mathbb{Z} \setminus \{-3,\cdots,1\}.$ |

| $\nu$ | $n$ | $\{B_{\nu,n}(k)\}$ |
|---|---|---|
| 3 | 4 | $B_{3,4}(-3)=\begin{bmatrix}-\frac{1}{32} & 0 & 0\\ -\frac{1}{128} & 0 & 0\\ -\frac{13}{384} & 0 & 0\end{bmatrix},\ B_{3,4}(-2)=\begin{bmatrix}\frac{5}{32} & 0 & 0\\ -\frac{1}{8} & 0 & 0\\ -\frac{13}{48} & 0 & 0\end{bmatrix},$ $B_{3,4}(-1)=\begin{bmatrix}-\frac{5}{16} & 0 & 0\\ -\frac{1}{384} & \frac{1}{2} & 0\\ -\frac{7}{384} & 0 & \frac{1}{2}\end{bmatrix},\ B_{3,4}(0)=\begin{bmatrix}\frac{5}{16} & 0 & 0\\ -\frac{1}{24} & 0 & 0\\ -\frac{7}{48} & 0 & 0\end{bmatrix},$ $B_{3,4}(1)=\begin{bmatrix}-\frac{5}{32} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ B_{3,4}(2)=\begin{bmatrix}\frac{1}{32} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$ $B_{3,4}(k)=O,\ k\in\mathbb{Z}\setminus\{-3,\cdots,2\}.$ |
| 3 | 5 | $B_{3,5}(-5)=\begin{bmatrix}-\frac{1}{64} & 0 & \\ -\frac{5}{2048} & 0 & \\ -\frac{77}{6144} & 0 & 0\end{bmatrix},\ B_{3,5}(-4)=\begin{bmatrix}\frac{3}{32} & 0 & 0\\ -\frac{5}{64} & 0 & 0\\ -\frac{77}{384} & 0 & 0\end{bmatrix},$ $B_{3,5}(-3)=\begin{bmatrix}-\frac{15}{64} & 0 & 0\\ -\frac{5}{3072} & 0 & 0\\ -\frac{13}{1024} & 0 & 0\end{bmatrix},\ B_{3,5}(-2)=\begin{bmatrix}\frac{5}{16} & 0 & 0\\ -\frac{5}{96} & 0 & 0\\ -\frac{13}{64} & 0 & 0\end{bmatrix},$ $B_{3,5}(-1)=\begin{bmatrix}-\frac{15}{64} & 0 & 0\\ -\frac{1}{10240} & \frac{1}{2} & 0\\ -\frac{5}{6144} & 0 & \frac{1}{2}\end{bmatrix},\ B_{3,5}(0)=\begin{bmatrix}\frac{3}{32} & 0 & 0\\ -\frac{1}{320} & 0 & 0\\ -\frac{5}{384} & 0 & 0\end{bmatrix},$ $B_{3,5}(1)=\begin{bmatrix}-\frac{1}{64} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad B_{3,5}(k)=O,\ k\in\mathbb{Z}\setminus\{-5,\cdots,1\}.$ |

In this chapter, we focused on multi-wavelet construction, following the wavelet construction method in [3]. In this work, the concept of multi-resolution analysis is applied to a general space decomposition, i.e. with no orthogonality (or bi-orthogonality) condition to be imposed. In fact, given a refinable vector function with matrix linearly independent integer shifts, Theorem 5.1.3 leads us to solving a system of matrix Laurent polynomial identities in order to ensure that condition (1.1.1) is obtained, with optimally short matrix filters $\{A(k)\}$ and $\{B(k)\}$. This then yields the construction of corresponding multi-wavelets having minimally short supports. In Sections 5.2 and 5.3, as an application of Theorem 5.1.3, such multi-wavelets were produced explicitly for, respectively, the Hermite refinable vector splines on $[-1, 1]$ and the refinable vector splines given in Chapter 3.

# Chapter 6

# CONCLUSIONS

In this thesis we have extended the (scalar) wavelet construction method of [3, Chapter 9] to the vector setting. Our approach is based on general (not necessarily orthogonal) refinement space decomposition to obtain multi-wavelets for the decomposition into low frequency and high frequency components of a given vector-valued signal.

As main building blocks we use refinable vector functions and their corresponding matrix refinement sequences, as was introduced in Chapter 1, with particular focus in Chapters 2, 3 and 4 on three different classes of arbitrarily smooth refinable vector splines.

The first such class of refinable vector splines, as developed in Chapter 2, was generated recursively by means of iterated vector convolution, from a starting non-continuous refinable vector spline built from Bernstein polynomials, and thereby extending a previous example of non-continuous refinability in [31] (see also [30]). However, this class of vector Bernstein convolutions lacks, from the continuous case upwards, the properties of integer-shift linear independence and stability, as is desirable in multi-wavelet construction. Nevertheless, as is evident from preliminary work in [30], useful applications in vector subdivision are possible.

Our second class of refinable vector splines, as presented in Chapter 3, and a special non-smooth case of which was first presented in [31] as a Fourier transform construction, before relying on the inverse Fourier transform to calculate the vector spline. We apply here the (more advantageous) reverse procedure of first applying a classical truncated power method to explicitly construct an arbitrarily smooth extension of the vector spline in [31], before only then rely on Fourier transforms to establish refinability, linear independence and stability, as well as explicitly obtaining the corresponding matrix refinement sequences.

As a third class of refinable refinable vector splines, we present in Chapter 4 the well-known refinable Hermite vector splines (see e.g. [32], [33], [34]), with their properties of interpolation and symmetry (or anti-symmetry). By applying techniques from polynomial algebra, we derive explicit and recursive formulations for efficient computation. Once again, we demonstrated that also this class possesses the properties of integer-shift linear independence and stability.

Finally, in Chapter 5, we extended the (scalar) wavelet construction method in [3] to characterize, for a given refinable vector function with linearly independent integer-shifts, a corresponding multi-wavelet and its matrix decomposition filter sequences in terms of a system of matrix Laurent polynomial identities. This method was then applied to the refinable vector splines of, respectively, Chapters 4 and 3, and thereby obtaining explicit formulations of shortest possible matrix decomposition filters, as well as, for these optimal choices, a corresponding multi-wavelet of minimum support.

In future research, we propose to pursue the following:

- As further development of the preliminary work in [30], the analysis with respect to vector subdivision of the refinable vector spline $\mathbf{\Phi}_m^{[\nu]}$ of Chapter 2, and its matrix refinement sequence $\left\{P_m^{[\nu]}(k)\right\}$.

- The investigation of integer-shift linear independence and $l^2$-stability of multi-wavelets $\mathbf{\Psi}_{\mathbf{\Phi},Q}$ as in Theorem 5.1.4.

- For any given refinable vector function with linearly independent integer-shifts, a comparison between the efficiency with respect to a signal decomposition of our multi-wavelets and those obtained from other methods, for example the CBC method.

- In our vector spline applications of Sections 5.2 and 5.3, the matrix Laurent polynomial identity systems of Theorem 5.1.3 could be solved in a direct manner, by virtue of the relatively simple structures of the Laurent matrix polynomial refinement symbols $\mathcal{P}_\nu^H$ and $\mathcal{P}_\nu$. In future envisaged research based on more complicated such refinement symbols $\mathcal{P}$, it would be helpful to first obtain an understanding of polynomials with matrix coefficients (see e.g. [43], [44]), and in particular the factorisation thereof (see e.g. [45]).

- Industrial applications of multi-wavelet decomposition techniques.

- The extension to general integer dilation $d \geq 2$ of this thesis.

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